23. The $W^{k,p}$-continuity of Wave Operators for Schrödinger Operators

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1. Introduction. Theorems. For Schrödinger operators $H = H_0 + V(x)$ and $H_0 = D_1^2 + \cdots + D_m^2$, $D_j = -i\partial/\partial x_j$, the wave operators $W_\pm$ and $Z_\pm$ are defined by the limits in $L^2 \equiv L^2(R^m)$:

$$W_\pm u = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}u, \quad Z_\pm u = \lim_{t\to\pm\infty} e^{itH_0}e^{-itH}P_c(H)u, \tag{1.1}$$

where $P_c(H)$ is the orthogonal projection onto the continuous spectral subspace $L_c^2(H)$ for H. We assume that $V(x)$ satisfies the following condition, where $m_* = (m-1)/(m-2)$, $\langle x\rangle = (1+|x|^2)^{1/2}$ and $\mathcal{F}$ is the Fourier transform. We take and fix $\sigma > 2/m_*$, $\delta > \max(m+2, 3m/2-2)$ and an integer $l \geq 0$.

Assumption 1.1. *$V(x)$ is a real valued function on R^m, $m \geq 3$, such that $\mathcal{F}(\langle x\rangle^\sigma D_x^\alpha V) \in L^{m_*}$ for any $|\alpha| \leq l$ and satisfies either* (1) $\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L_{m*}} \equiv C(V)$ *is sufficiently small or* (2) $m = 2m'-1$ *is odd and* $|D^\alpha V(x)| \leq C_\alpha\langle x\rangle^{-\delta}$ *for any* $|\alpha| \leq \max\{l, m'-4+l\}$.

Under the assumption, V is H_0-bounded and is short-range in the sense of Agmon [1]. Hence H with domain $D(H) = D(H_0) = W^{2,2}$ is selfadjoint and both limits in (1.1) exist ([1], [8]); $W_\pm$ are partial isometries from L^2 onto $L_c^2(H)$ and $Z_\pm = W_\pm^*$. Consequently, the continuous part H_c of H is unitarily equivalent to H_0 and, for any Borel function f, $f(H)P_c(H) = W_\pm f(H_0)W_\pm^*$, $f(H_0) = W_\pm^* f(H)\,P_c(H)W_\pm$. The main result of this paper is the following

Theorem 1.1. *Let V satisfy Assumption 1.1 and let 0 be neither eigenvalue nor resonance of H. Then, for any $1 \leq p \leq \infty$ and integral $0 \leq k \leq l$, $W_\pm$ and $Z_\pm$ originally defined on $L^2 \cap W^{k,p}$ can be extended to bounded operators in $W^{k,p}$.*

Remark 1.1. We say 0 is resonance of H if $-\Delta u(x) + V(x)u(x) = 0$ has a solution u such that $\langle x\rangle^{-\gamma}u(x) \in L^2$ for any $\gamma > 1/2$ but not for $\gamma = 0$. Under the assumption, 0 is not resonance if $m \geq 5$, and is neither eigenvalue nor resonance if $C(V)$ is small enough.

Remark 1.2. If 0 is resonance, Theorem 1.1 never holds. If 0 is eigenvalue of H, then it does not hold in general. This can be seen by comparing the results of [3] or [9] with Theorem 1.3 below.

In the sequel, we always assume that the condition of Theorem 1.1 is satisfied. For Banach spaces X and Y, $B(X, Y)$ is the space of bounded operators from X to Y, $B(X) = B(X, X)$. Theorem 1.1 yields the following

Theorem 1.2. *Let $1 \leq p, q \leq \infty$ and let $0 \leq k, k' \leq l$ be integers. Then*:

$$C^{-1}\|f(H_0)\|_{B(W^{k,p},W^{k',q})} \leq \|f(H)P_c(H)\|_{B(W^{k,p},W^{k',q})} \leq C\|f(H_0)\|_{B(W^{k,p},W^{k',q})},$$