13. *Shifted Divisor Problem and Random Divisor Problem*

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In this note, we consider the following sum which coincides with the sum of the Dirichlet divisor problem (see [1], [4]) when $\alpha = \beta = 1$;

$$D(x\,;\alpha,\beta) = \sum_{\substack{(m+\alpha)(n+\beta)\le x\\ m,n\in N\cup\{0\}}}{}' 1, (0<\alpha\le 1,\ 0<\beta\le 1) \tag{1}$$

where the dash on Σ means that we count $\frac{1}{2}$ in place of 1 when $(m+\alpha)\cdot(n+\beta)=x$ in the sum. $D(x\,;\alpha,\beta)$ is the number of lattice points inside the region: $\{(u,v)\in \boldsymbol{R}^2 : 0<u,\ 0<v,\ uv\le x\}$ where the lattice points are shifted by α in u-direction and by β in v-direction from their original sites.

About this sum, we obtain the following theorems of which detailed proofs will be given elsewhere together with related results. We only give here outlines of the proofs. First, we have the following Voronoï-type identity:

Theorem 1 (Voronoï-type identity).

$$D(x\,;\alpha,\beta) = x\log x + \left\{\left(-\frac{\Gamma'}{\Gamma}(\alpha)\right) + \left(-\frac{\Gamma'}{\Gamma}(\beta)\right) - 1\right\}x + \left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}-\beta\right) + \Delta(x\,;\alpha,\beta), \tag{2}$$

$$\begin{aligned}\Delta(x\,;\alpha,\beta) = &-\frac{\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_{\alpha,\beta}(n)+d_{-\alpha,-\beta}(n)}{\sqrt{n}}Y_1(4\pi\sqrt{nx}) - \\ &-\frac{\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_{\alpha,-\beta}(n)+d_{-\alpha,\beta}(n)}{\sqrt{n}}K_1(4\pi\sqrt{nx}) - \\ &-i\frac{\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_{\alpha,-\beta}(n)-d_{-\alpha,\beta}(n)}{\sqrt{n}}J_1(4\pi\sqrt{nx}),\end{aligned} \tag{3}$$

where $\Gamma(\cdots)$ denotes the gamma-function and $Y_1(\cdots)$, $K_1(\cdots)$, $J_1(\cdots)$ denote the Bessel functions in ordinary sense:

$$\begin{aligned}Y_1(x) = &\frac{1}{\pi}\left(\frac{x}{2}\right)\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}\left(-\frac{x^2}{4}\right)^k\cdot \\ &\cdot\left\{2\log\left(\frac{x}{2}\right)-\psi(k+1)-\psi(k+2)\right\}-\frac{2}{\pi x},\end{aligned}$$

$$\begin{aligned}K_1(x) = &\left(\frac{x}{2}\right)\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}\left(\frac{x^2}{4}\right)^k\cdot \\ &\cdot\left\{\log\left(\frac{x}{2}\right)-\frac{1}{2}\psi(k+1)-\frac{1}{2}\psi(k+2)\right\}+\frac{1}{x},\end{aligned}$$

$$\left(\psi(x) = \frac{\Gamma'}{\Gamma}(x)\right),$$