96. On the Rank of an Elliptic Curve in Elementary 2-extensions

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1. Let E be an elliptic curve (i.e., an abelian variety of dimension one) defined over an algebraic number field k. For any finite field extension K of k, we denote by $E(K)$ the group of K-rational points of E. We define *the Mordell-Weil rank over K of E by*

$$\mathrm{rank}(E\,;K) = \dim_{\boldsymbol{Q}} E(K) \otimes_{\boldsymbol{Z}} \boldsymbol{Q},$$

which is known to be finite.

The extension K/k is called an *elementary* 2-*extension* if it is a (Galois) (pro-) 2-extension with the Galois group of exponent 2.

This note grew out of an effort to generalize Ono's theorem [7] on the relative Mordell-Weil rank (his $E(\kappa)$ is our E_{κ}) and its aim is to construct elliptic curves whose Mordell-Weil rank becomes infinite in a tower of elementary 2-extensions.

We should note here that Kurčanov ([4],[5]) constructed elliptic curves defined over $\boldsymbol{Q}$ whose ranks are infinite or stable in a $\boldsymbol{Z}_p$-extension based on the theory of Mazur.

2. Let k be an algebraic number field and suppose we are given a (finite or infinite) subset $\Sigma = \{d_\lambda\}_{\lambda\in\Lambda}$ of $k^\times/(k^\times)^2$. We can assign a quadratic extension $k_\lambda = k(\sqrt{d_\lambda})$ to each d_λ in the set Σ.

For any non-empty finite subset S of Λ, we set

$$k_S = k\left(\sqrt{\prod_{i\in S} d_i}\right) \text{ and } k(S) = k(\{\sqrt{d_i} \mid i \in S\}).$$

We call the set Σ a *primitive set* if $[k(S):k] = 2^{\#S}$ holds for all finite subsets S of Σ. If Σ is primitive, then the fields k_T's ($T \neq \phi$, $T \subseteq S$) are exactly $2^{\#S} - 1$ different quadratic extensions over k in $k(S)$. For an elliptic curve E defined over k, we denote by E^S the twist of E by the quadratic character of k_S/k.

The following proposition is the key to our construction.

Proposition 1. *Suppose that $\Sigma = \{d_\lambda\}_{\lambda\in\Lambda}$ is primitive and let S be any finite subset of Λ. Then we have*

$$\mathrm{rank}(E\,;k(S)) = \sum_{T\subseteq S} \mathrm{rank}(E^T\,;k),$$

where the sum is taken for all subsets T of S.

Proof. Put $S = \{1,2,\dots,m\}$ and $S' = \{1,2,\dots,m-1\}$. When $m = 1$, the proposition is classical (for instance, see [1]). It is easy to see that $[k(S):k(S')] = 2$ and $k(S) = k(S')(\sqrt{d_m})$. Therefore we obtain

$$\mathrm{rank}(E\,;k(S)) = \mathrm{rank}(E\,;k(S')) + \mathrm{rank}(E^{\{m\}}\,;k(S'))$$