93. A New Formula of Arc Length in Euclidean Space and its Application

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Abstract: We shall introduce a new formula of the arc length of a rectifiable curve in the Euclidean space. By this new formula, the arc length is represented as a supremum of a linear functional on a subset of continuous functions defined on an interval of the real line, in which the parameter of the curve runs. This linearity enables us to calculate the arc length of admissible curves introduced in [1].

Key word: arc length.

1. New formula. Let $x=(x_1,\dots,x_n)$ be an orthonormal coordinate system of $\boldsymbol{R}^n$. Let Λ be a continuous curve parametrized by θ, $0 \leqq \theta \leqq 2\pi$ and given by the equations $x_j = x_j(\theta)$, $j=1,\dots,n$. By the points $0=\theta_0, \theta_1,\dots,\theta_N = 2\pi$ we divide up the interval $[0, 2\pi]$ into N sub-intervals $I_k = [\theta_{k-1}, \theta_k)$, $k=1,\dots,N$ of lengths $|I_k|$, $k=1,\dots,N$. We denote this division by Δ. Then the length $|\Lambda|$ of Λ is defined as

$$|\Lambda| = \sup_{\Delta} \left\{ \sum_{k=1}^{N} \Big(\sum_{j=1}^{n} (x_j(\theta_k) - x_j(\theta_{k-1}))^2 \Big)^{1/2} \right\}. \tag{1.1}$$

If $|\Lambda|$ exists, the curve Λ is said to be *rectifiable*. It is well known that if Λ is rectifiable then $x_j(\theta)$, $j=1,\dots,n$ are continuous functions of bounded variation and therefore $\dot{x}_j$, $j=1,\dots,n$, derivatives of $x_j(\theta)$, $j=1,\dots,n$, in the sense of distributions of L. Schwartz are Radon measures of atom-free (having no point mass), that is, for every θ in $[0, 2\pi]$, $\dot{x}_j(\{\theta\})=0$, $j=1,\dots,n$. Hence $\dot{x}_j([\theta_{k-1}, \theta_k]) = \dot{x}_j([\theta_{k-1}, \theta_k))$, $j=1,\dots,n$, $k=1,\dots,N$. Thus we can regard $\dot{x}_j$, $j=1,\dots,n$ as continuous linear forms over $C([0, 2\pi])$, the space of real-valued continuous functions on $[0, 2\pi]$. We denote these continuous forms by $\dot{x}_j[\phi]$, $j=1,\dots,n$ for ϕ in $C([0, 2\pi])$ and these forms are also defined over the space of step functions. Then we have the following new formula of the arc length of a rectifiable curve:

Theorem 1.1.

$$\begin{aligned} |\Lambda| &= \sup_{\sum_{j=1}^n \phi_j^2 \leqq 1,\, \phi_j \in C([0,2\pi])} \Big(\sum_{j=1}^{n} \dot{x}_j[\phi_j] \Big) \\ &= \sup_{\sum_{j=1}^n \phi_j^2 \leqq 1,\, \phi_j \in C([0,2\pi])} \langle (\dot{x}_1,\dots,\dot{x}_n), (\phi_1,\dots,\phi_n) \rangle . \end{aligned} \tag{1.2}$$

Remark. Obviously $C([0, 2\pi])$ can be replaced by $C^\infty([0, 2\pi])$.

Proof. Denote the right-hand side of (1.1) by A and the right-hand side of (1.2) by B.

First we shall show $A \leqq B$. Put