# 90. *Some Examples of Global Gevrey Hypoellipticity and Solvability*

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**1. Notations and results.** Let $T^2 := R^2/Z^2$ be the two dimensional torus, where $R$ and $Z$ are the sets of real numbers and integers respectively. We denote the variables in $T^2$ by $(x, y)$ and the differentiations on $T^2$ by $\partial_x = \partial/\partial x$, and $\partial y = \partial/\partial y$. We denote by $C^\infty(T^2)$ the set of smooth functions on $T^2$. For $\sigma \geq 1$ we say that a function $f(x, y) \in C^\infty(T^2)$ belongs to the Gevrey class $G^\sigma(T^2)$ if for some $C > 0$

(1.1) $\quad |\partial_x^m \partial_y^n f(x, y)| \leq C^{m+n+1}(m!n!)^\sigma$, for all $m, n \in N$, $(x, y) \in T^2$,

with the convention that $G^\infty(T^2) := C^\infty(T^2)$, if $\sigma = \infty$. We denote by $G^\sigma(T^2)'$ the space of ultradistributions of class $\sigma$ on $T^2$. Clearly, $G^1(T^2)$ is the set of analytic functions on $T^2$ and $G^1(T^2)'$ coincides with the class of periodic hyperfunctions on $T^2$ (cf. [6] and [9]).

A differential operator $P$ is said to be globally $G^\sigma(T^2)$ solvable on $T^2$ if for every $f \in G^\sigma(T^2)$ there exists an ultradistribution $u \in G^\sigma(T^2)'$ satisfying $Pu = f$. We say that $P$ is globally $G^\sigma(T^2)$ hypoelliptic if $u \in G^\sigma(T^2)$ when $Pu \in G^\sigma(T^2)$ and $u \in G^\sigma(T^2)'$. The operator $P$ is said to be locally $G^\sigma$ solvable at a point $p \in T^2$ if there exists a neighborhood $U$ of $p$ such that for every $f \in G_0^\sigma(U)$, there exists an ultradistribution $u \in G^\sigma(U)'$ such that $Pu = f$ in $U$. Similarly, we say that $P$ is locally $G^\sigma$ hypoelliptic at $p$ if the following condition holds; if a point $p$ does not belong to $G^\sigma$ singular support of $Pu$ then $p$ does not belong to $G^\sigma$ singular support of $u$.

In this note we shall give examples of first order operators with real coefficients on tori whose global properties are exotic in the following sense: Their global hypoellipticity and solvability in Gevrey class depend on Gevrey index $\sigma$. This makes a clear contrast to the known local results for operators of real principal type (cf. [5] and[1]). In fact, the first order analytic pseudodifferential operators of real principal type are not locally $G^\sigma$ hypoelliptic for any $1 \leq \sigma \leq \infty$ and they are locally $G^\sigma$ solvable for all $1 \leq \sigma \leq \infty$ (cf. [5] and [9]). In the global case, we have the following

**Theorem 1** (Global hypoellipticity). *For every number $\sigma$, $1 \leq \sigma < \infty$ we can find infinitely many linearly independent real-valued functions $a \in G^1(T)$ such that the operators $P = \partial_x - a(x)\partial_y$ are globally $G^\theta(T^2)$ hypoelliptic if $1 \leq \theta \leq \sigma$, while they are not globally $G^\theta(T^2)$ hypoelliptic if $\sigma < \theta \leq \infty$.*

**Theorem 2** (Global solvability). *For every number $\sigma$, $1 \leq \sigma < \infty$ we can*

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