7. A Note on Class Number One Problem for Real Quadratic Fields

By Hideo YOKOI

Graduate School of Human Informatics, Nagoya University

(Communicated by Shokichi IYANAGA, M. J. A., Jan. 12, 1993)

In our previous paper[2], for the fundamental unit ε_p of the real quadratic field $\boldsymbol{Q}(\sqrt{p})$ of prime discriminant, we showed that there exist exactly 30 real quadratic fields $\boldsymbol{Q}(\sqrt{p})$ of class number one satisfying $\varepsilon_p < 2p$ with one more possible exception of prime discriminant p.

On the other hand, in the paper [3], for a positive square-free integer D we defined new D-invariants m_D, n_D, and using them we provided some estimate formulas of the class number and the fundamental unit of the real qnadratic field $\boldsymbol{Q}(\sqrt{D})$.

In this paper, using one of those estimate formulas of the class number we shall provide a kind of improvement of Theorem 2 in [2], which relates to class number one problem for real quadratic fields.[1)]

For any positive square-free integer D, we denote by h_D and by

$$\varepsilon_D = (t_D + u_D\sqrt{D})/2 \ (>1)$$

the class number and the fundamental unit of the real quadratic field $\boldsymbol{Q}(\sqrt{D})$ respectively, and put

$$\boldsymbol{D}_- = \{D\colon \text{positive square-free integer with } N\varepsilon_D = -1\}.$$

Our main purpose of this paper is to prove the following theorem:

Theorem. For arbitrarily chosen and fixed natural number h_0 and real number c greater than 2, there exists only a finite number of real quadratic fields $\boldsymbol{Q}(\sqrt{D})$ $(D \in \boldsymbol{D}_-)$ such that

(1) $$\varepsilon_D < D(e^{D^{\frac{1}{c}}} - 1) \quad \text{or} \quad t_D < D(e^{D^{\frac{1}{c}}} - 1),$$

and

(2) $$h_D \leqq h_0.$$

To prove this theorem, we need several lemmas.

Lemma 1. For any $D > 5$ in $\boldsymbol{D}_-$,

$$[t_D/D] = [\varepsilon_D/D] = [u_D^2/t_D]$$

holds, where $[x]$ means the greatest integer less than or equal to x.

For the proof, see Theorems 2.1, 2.3 and their proofs in [3].

Here, if we put

$$m_D = [t_D/D] \ (= [\varepsilon_D/D])$$

the same as in [3], then we have easily the following lemma:

Lemma 2. If $s \geqq 11.2$ and $D \geqq e^s$ for D in $\boldsymbol{D}_-$, then

$$h_D > 0.3275 \cdot s^{-1} \cdot D^{(s-2)/(2s)} / \{\log D(m_D + 1)\}$$

holds with one possible exception of D.

For the proof, see Theorem 2.3 in [3].

---

1) Cf. H. Yokoi [1].