42. *Fully Idempotent Semirings*

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(Communicated by Shokichi IYANAGA, M. J. A., June 8, 1993)

In this paper A will denote a semiring $(A, +, \cdot)$ as defined, for example, in [2], that is, two semigroups $(A, +)$ and $(A, \cdot)$ such that addition distributes over multiplication. Moreover, we shall always assume that $(A, +)$ is commutative and $(A, +, \cdot)$ has an absorbing zero 0, that is, $a + 0 = 0 + a = a$ and $0 \cdot a = a \cdot 0 = 0$ hold for all $a \in A$. The notions of left, right, and two-sided ideals, as well as sums and products of such ideals are defined as usual. The word ideal will always mean a two-sided ideal. An ideal P of A is called *prime* (*irreducible*; *strongly irreducible*) if $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P (I \cap J = P \Rightarrow I = P$ or $J = P$; $I \cap J \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P)$ holds for all ideals I, J of A. Thus any prime ideal is strongly irreducible and any strongly irreducible ideal is irreducible. A semiring A is called *fully idempotent* if each (two-sided) ideal of A is idempotent (an ideal I is idempotent if $I^2 = I$). A semiring A is called (*von Neumann*) *regular* if $x \in xAx$, for all $x \in A$ (cf. [8,10]). Regular semirings and simple semirings (i. e. having no non-zero proper ideals) form proper subclasses of fully idempotent semirings. Below we characterize fully idempotent semirings by the property that each ideal is the intersection of those prime ideals which contain it. We also obtain a similar characterization of semisimple semigroups, that is, semigroups all of whose ideals are idempotent.

We begin with the following result which is due to Courter [4]. Courter, in fact, proved this result for rings instead of semirings. However, an examination of his proof shows that it works in the case of semirings.

Proposition 1. *The following assertions for a semiring A are equivalent*:

1. *A is fully idempotent*;
2. *for each pair of ideals I, J of A, $I \cap J = IJ$*;
3. *for each right ideal R and two-sided ideal I, $R \cap I \subseteq IR$*;
4. *for each left ideal L and two-sided ideal I, $L \cap I \subseteq LI$.*

Recall that the lattice of ideals of a semiring is not, in general, distributive or even modular (cf. [7]). Below, we show that the ideal lattice of a fully idempotent semiring is a complete Brouwerian and hence distributive lattice. A lattice $\mathcal{L}$ is called *Brouwerian* if, for any $a, b \in \mathcal{L}$, the set of all $x \in \mathcal{L}$ satisfying $a \wedge x \leq b$ contains a greatest element c, the *pseudo-complement* of a relative to b.

Proposition 2. *If A is a fully idempotent semiring, then the ideal lattice $\mathcal{L}_A$ of A is a complete Brouwerian lattice.*

Proof. Clearly, $\mathcal{L}_A$ is a complete lattice under the sum and intersection