41. *Center Curves in the Moduli Space of the Real Cubic Maps*

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1. Center curves. We consider the family of real cubic maps $x \mapsto g(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ ($c_3 \neq 0$, $c_i \in \boldsymbol{R}$). By a suitable real affine transformation, any map $g(x)$ is transformed to a unique map $f(x) = \sigma x^3 - 3Ax + \sqrt{|B|}$, where $\sigma := sgn(g''')$. The real affine conjugacy class of g or f can be represented by (A, B) if $B \neq 0$. But if $B = 0$, σ should be added as an essential class invariant, as $x \mapsto x^3 - 3Ax$ and $x \mapsto -x^3 - 3Ax$ belong to different classes. Milnor ([1]) defined thus the disjoint union of the upper half-plane $\boldsymbol{H}^+ = \{(A, B) \mid B \geq 0\}$ and the lower half-plane $\boldsymbol{H}^- = \{(A, B) \mid B \leq 0\}$ to be the **moduli space** of the conjugacy classes of our maps.

The map $x \mapsto f(x)$ has two critical points $\pm\sqrt{\sigma A}$ (which may coincide or be purely imaginary) which will be denoted with p_1, p_2. When the orbit $\{f^n(p_1), f^n(p_2); n = 1,2,\ldots\}$ is a finite set, f is called a **center map** and the coordinates (A, B) of f will be called a **center** in the moduli space.

Following Milnor ([1]), the centers are classified as follows. (In the following t, p, q denote integers.)

A center is of the type $\mathcal{A}_p$ if two critical points of the center map coincide $p_1 = p_2$ and has the period $p : f^p(p_1) = p_1$. (In fact, only possible values for p in this case are 1, 2.) A center is of the type $\mathcal{B}_{p+q}$ if $f^p(p_1) = p_2$ and $f^q(p_2) = p_1$; of the type $\mathcal{C}_{(t)q}$ if $f^t(p_1) = p_2$ and $f^q(p_2) = p_2$; of the type $\mathcal{D}_{p,q}$ if $f^p(p_1) = p_1$ and $f^q(p_2) = p_2$.

These exhaust all types of centers. It is clear that there are only a finite number of centers of a given type.

Example. There exist three centers of type $\mathscr{C}_{(3)1}$. The corresponding parameters are $(A, B) = (-.75040, -.18820)$, $(-.74949, -.18679)$, $(-.0924912, -.0614376)$.

From now on, we shall limit our consideration to the case $\sigma A > 0$. Then we observe that the following theorem holds.

Theorem. *For a given p, there exist an algebraic curve CDp containing all centers of the type $\mathcal{C}_{(k)p}$ and $\mathcal{D}_{k,p}$, and another algebraic curve BCp containing all centers of the type $\mathcal{B}_{p+k}$ and $\mathcal{C}_{(p)k}$. Precisely we obtain the following curves;*

$$\mathrm{CD1} : B = 4A\left(A + \frac{1}{2}\right)^2,$$

$$\mathrm{BC1} : B = 4A\left(A - \frac{1}{2}\right)^2,$$

$$\mathrm{CD2} : B^2 - 8A^3B + 4A^2B - 5AB + 2B + 16A^6 - 16A^5 - 12A^4 + 16A^3 - 4A + 1 = 0,$$