40. *All Congruent Numbers less than 10000*

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1. Notations and result. If n is a square free positive integer such that

$$n = xy/2, \quad x^2 + y^2 = z^2$$

for some rational numbers x, y, z, then n is said to be a congruent number. If $x/y = u^2$ for some rational number u, we write $x \simeq y$. Because of the well-known expressions of Pythagorean numbers, it is clear that n is a congruent number if and only if

(1) $$n \simeq ab(a-b)(a+b)$$

for some positive integers a, b, $0 < b < a$. If

(2) $$n \simeq cd(c^2 + d^2)/2$$

for some positive integers c, d, $0 < d < c$, then

$$n \simeq (c+d)^2 (c-d)^2 \{(c+d)^2 - (c-d)^2\}\{(c+d)^2 + (c-d)^2\}.$$

Therefore n is a congruent number (cf. [3], [10]).

Now, let E_n be the elliptic curve

$$E_n : y^2 = x^3 - n^2 x.$$

It is known that n is a congruent number if and only if the rank $r(E_n)$ of E_n is positive (cf. [4], [7]). It was conjectured by Birch and Swinnerton-Dyer [1] that $L(E_n, 1) = 0$ is equivalent with $r(E_n) > 0$ where $L(E_n, s)$ is the Hasse-Weil L-function of E_n. It was proved, furthermore, by [5], that if $r(E_n) > 0$ then $L(E_n, 1) = 0$, and Tunnell's theorem [11] permits us easily to determine if $L(E_n, 1) = 0$ or not for given n. Using computer we got the next result.

Main result. Let n be a square free positive integer less than 10000, then

$$n = \text{a congruent number} \Longleftrightarrow L(E_n, 1) = 0.$$

More precisely, if $n \equiv 5,6,7 \pmod 8$ then n is a congruent number and if $n \equiv 1,2,3 \pmod 8$ then all such congruent numbers are listed in Table II.

2. Methods. If $n \equiv 5,6,7 \pmod 8$, then we have $L(E_n, 1) = 0$ (this can be shown without using [11]. Cf. [7]). In [6] it was proved that if $L(E_n, 1) = 0$, $L'(E_n, 1) \neq 0$ then $r(E_n) > 0$. We computed $L'(E_n, 1)$ in the range $n \equiv 5,6,7 \pmod 8$, $n < 10000$(cf. [2]). We got $L'(E_n, 1) \neq 0$ except for next 15 numbers: $n =$ 1254,2605,2774,3502,4199,4669,4895,6286, 6671,7230,7766,8005,9015,9430,9654. For these 15 numbers we got $L^{(3)}(E_n, 1) \neq 0$ (cf. [2]) and we could easily find the solutions of (1) in the range $0 < b < a \leq 1601$ (cf. [9]). Therefore if $n \equiv 5,6,7 \pmod 8$ and $n < 10000$, then n is a congruent number. There are 3050 such numbers.

Using [11] we got 453 numbers such that $n \equiv 1,2,3 \pmod 8$, $n < 10000$, $L(E_n, 1) = 0$. For these 453 numbers we can find solutions of