39. *On the Measure on the Set of Positive Integers*

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R. C. Buck [1] constructed as follows a measure μ on the set $\boldsymbol{N} = \{0,1,2,\ldots\}$ of positive integers. For an arithmetic progression $A = \{an + b \mid n \in \boldsymbol{N}\} = a\boldsymbol{N} + b$, $a, b \in \boldsymbol{N}$, $a \neq 0$, put $\mu(A) = a^{-1}$. Let $\boldsymbol{A}$ be the class of all arithmetic progressions and $\boldsymbol{B}$ the class of subsets of $\boldsymbol{N}$ which are finite disjoint unions of elements of $\boldsymbol{A}$; thus if $B \in \boldsymbol{B}$, then $B = \sum_{i=1}^{k} A_i$ (disjoint union) $A_i \in \boldsymbol{A}$, $i = 1,2,\ldots,k$. For such B, put $\mu(B) = \sum_{i=1}^{k} \mu(A_i)$. For any subset C of $\boldsymbol{N}$, $\mu(C)$ is defined to be inf $\mu(B)$, $B \in \boldsymbol{B}$ and $C \subset B \cup F$, where F is a finite subset of $\boldsymbol{N}$.

On the other hand, we have another measure ν on $\boldsymbol{N}$, used by J.-L. Mauclaire [2] to obtain various results. Let P denote the set of all prime numbers. For $p \in P$, the additive group $\boldsymbol{Z}_p$ of p-adic integers with p-adic topology is a compact abelian group, which has therefore the Haar measure ν_p with $\nu_p(\boldsymbol{Z}_p) = 1$. The product group $G = \Pi_{p\in P} \boldsymbol{Z}_p$ with the product topology is again a compact abelian group with product measure $\nu = \Pi_{p\in P} \nu_p$. $\boldsymbol{Z}$ is considered as a dense subgroup in G, and $\boldsymbol{N}$ as an open and closed subset of $\boldsymbol{Z}$ which is also dense in G.

J.-L. Mauclaire [3] discussed the relationship between μ and ν using Riemann-Stieltjes integration. In this note, we shall show that this relationship can be directly clarified using only topological considerations.

Remark. The above introduced notations $\boldsymbol{A}$, $\boldsymbol{B}$, μ, ν, ν_p will be used throughout this note in the same meanings. Let us recall that $U_p(x, e) = x + p^e\boldsymbol{Z}_p$, $x \in \boldsymbol{Z}_p$, $e \in \boldsymbol{N}$, constitute an open basis of $\boldsymbol{Z}_p$ and $V_S(U_p(x_p, e_p)) = \Pi_{p\in S} U_p(x_p, e_p) \times \Pi_{q\in P-S} \boldsymbol{Z}_q$ where S runs over the finite subset of P, $x_p \in \boldsymbol{Z}_p$, $e_p \in \boldsymbol{N}$, an open basis of G. For a subset M of G, $\overline{M}$ will denote the closure of M in G. Recall, furthermore, that $\nu_p(U_p(x, e)) = \nu_p(x + p^e\boldsymbol{Z}_p)$ dose not depend on x and is equal to p^{-e}, so that $\nu(V_S(U_p(x_p, e_p))) = \Pi_{p\in S} p^{-e_p}$.

Our main result will follow from the following two propositions.

Proposition 1. *For any open and closed non-empty subset O in G, $O \cap \boldsymbol{N}$ belongs to $\boldsymbol{B}$.*

Proof. O is a union of sets of form $V_S(U_p(x_p, e_p))$, because O is open. As G is compact, O is also compact. So O is a finite union of $V_S(U_p(x_p, e_p))$. Now $V_S(U_p(x_p, e_p)) \cap \boldsymbol{N} = a\boldsymbol{N} + b \in \boldsymbol{A}$ where $a = \Pi_{p\in P} p^{e_p}$ and $b \equiv x_p \bmod p^{e_p}$, so that $O \cap \boldsymbol{N}$ is an element of $\boldsymbol{B}$.

Proposition 2. *For $B \in \boldsymbol{B}$, $\overline{B}$ is an open and closed subset of G, and $\mu(B) = \nu(\overline{B})$.*

Proof. For $A = a\boldsymbol{N} + b \in \boldsymbol{A}$, we set $a = \Pi_{p\in P} p^{e_p}$ where S is a finite