38. A Remark to the Paper "On the Stabilizer of Companion Matrices" by J. Gomez-Calderon

By You ASAEDA

Department of Mathematics, Gakushuin University

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In the paper [1] cited in the title, the following question is treated.

Let R be a commutative ring with 1, M the ring of $n \times n$ matrices over R, $n \geq 2$, $f(X) = X^n - \sum_{i=0}^{n-1} b_i X^i$ a polynomial of degree n in $R[X]$, $C(f)$ the companion matrix of $f(X)$ defined by

$$C(f) = \begin{pmatrix} 0 & \cdots & \cdots & 0 & b_0 \\ 1 & \ddots & & \vdots & b_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & b_{n-1} \end{pmatrix}$$

(which has $f(X)$ as the characteristic polynomial). An element A of M such that

$$(*) \qquad AC(f) = C(f)A$$

is called a stabilizer of $f(X)$, the set of which will be denoted by $S(f)$. In [1], a characterization of $A \in S(f)$ is given, and as an application, the following result is proved:

If R is a finite field F and $f(X)$ is irreducible, then every non-zero element of $S(f)$ is invertible.

The proof of this fact given in [1] is based on [2] and uses essentially the finiteness of F. In this note, another characterization of $S(f)$ will be given (Theorem 1) and it will be proved that the above proposition holds for any field F (Theorem 2).

The above notations R, M, $C(f)$, $S(f)$ will be used in the same meanings throughout this note.

Theorem 1. *$\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ being n column vectors $\in \boldsymbol{R}^n$, the follwing four conditions on $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) \in M$ are mutually equivalent.*

(1) $$AC(f) = C(f)A, \qquad i.e. \quad A \in S(f),$$

(2) $$\boldsymbol{a}_{i+1} = C(f)\boldsymbol{a}_i, \qquad i = 1,2,\ldots, n-1,$$

(3) $$\boldsymbol{a}_i = C(f)^{i-1}\boldsymbol{a}_1, \qquad i = 1,2,\ldots, n,$$

(4) *A can be expressed as $g(C(f))$, $g(X) \in R[X]$, $\deg g(X) \leq n-1$.*

Proof. In computing both sides of $(*)$ for $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ we obtain

$$(\boldsymbol{a}_2, \ldots, \boldsymbol{a}_n, \sum_{i=0}^{n-1} b_i \boldsymbol{a}_{i+1}) = (C(f)\boldsymbol{a}_1, \ldots, C(f)\boldsymbol{a}_n).$$

Comparing the first $n-1$ columns, we see

$$\boldsymbol{a}_{i+1} = C(f)\boldsymbol{a}_i, \quad i = 1,2,\ldots,n-1.$$