37. *Discrete Mean Values of Hurwitz Zeta-functions*

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1. The results. Let $\zeta(s, \alpha)$ be the Hurwitz zeta-function with a positive parameter α, and $\zeta_1(s, \alpha) = \zeta(s, \alpha) - \alpha^{-s}$. The behaviour of the integral

$$I(t) = \int_0^1 \left| \zeta_1\left(\frac{1}{2} + it, \alpha\right) \right|^2 d\alpha$$

has been studied by various authors. Zhang [5] [8] conjectured that for any $t \geq 1$,

$$I(t) = \log(t/2\pi) + \gamma + O(t^{-1/4}), \tag{1.1}$$

where γ is Euler's constant. (Perhaps this conjecture had been well-known among Indian number theorists.) Quite recently, Zhang [11] proved this conjecture; indeed, he has shown the following far better result:

$$I(t) = \log(t/2\pi) + \gamma - 2\mathrm{Re}\frac{\zeta\left(\frac{1}{2} + it\right)}{\frac{1}{2} + it} + O(t^{-1}), \tag{1.2}$$

where $\zeta(s)$ is the Riemann zeta-function.

Let q be a positive integer. In this note we consider the discrete mean value

$$J(s, q) = \sum_{1 \leq a \leq q} | \zeta(s, a/q) |^2.$$

Let $\Gamma(s)$ be the gamma-function, $\psi(s) = (\Gamma'/\Gamma)(s)$, N be a positive integer, and define

$$R_N(u, v; q) = \frac{1}{\Gamma(u)\Gamma(v)} \int_0^\infty \frac{y^{v+N-1}}{e^y - 1} \times$$
$$\times \int_0^\infty \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} h^{(N)}(x + q^{-1}\tau y) x^{u-1} \, d\tau \, dx dy$$

for $0 < \mathrm{Re}\, u < N + 1$ and $\mathrm{Re}\, v > -N + 1$, where $h^{(N)}(z)$ is the N-th derivative of

$$h(z) = \frac{e^z}{e^z - 1} - \frac{1}{z}.$$

Then we have

Theorem 1. *For any $t \geq 1$ and any positive integers N and q, we have*

(1.3) $J\left(\frac{1}{2} + it, q\right)$

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