36. A Note on the Rational Approximations to $\tanh\frac{1}{k}$

By Takeshi OKANO

Department of Mathematics, Saitama Institute of Technology
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§ 1. **Introduction.** I. Shiokawa [4] proved the following theorem.

Theorem A. *Let k be a positive integer. Then there is a positive constant C depending only on k such that*

$$\left|\tanh\frac{1}{k}-\frac{p}{q}\right|>C\frac{\log\log q}{q^2\log q}$$

for all integers p and q with $q\geq 3$.

The purpose of this note is to prove the following theorem which shows that constant C in Theorem A is an effectively computable number depending only on $k\geq 2$.

Theorem. *Let k and N be positive integers with $k\geq 2$ and $N\geq 10$, and let p_n/q_n be the n-th convergent of $\tanh\frac{1}{k}$. Let γ_N and δ_n be defined by*

$$\gamma_N=2\left(k+\frac{k+1}{N-1/2}\right)\left(1+\frac{\log\log(2k(N+1)/e)}{\log(N+1)}\right)$$

and

$$\delta_n=\frac{(k(2n+1)+2)\log\log q_n}{\log q_n},$$

respectively. Let γ be any constant such that

$$\gamma\geq\max\{\gamma_N,\ \gamma_N^*\},$$

where

$$\gamma_N^*=\max\{\delta_n\mid 1\leq n<N\}.$$

Then

$$\left|\tanh\frac{1}{k}-\frac{p}{q}\right|>\frac{\log\log q}{\gamma q^2\log q}$$

for all integers p and q with $q\geq 2$.

We now record two corollaries of the theorem.

Corollary 1. *For all integers p and q with $q\geq 2$,*

$$\left|\tanh\frac{1}{2}-\frac{p}{q}\right|>\frac{\log\log q}{6q^2\log q}.$$

Corollary 2. *For all integers p and q with $q\geq 2$,*

$$\left|\tanh\frac{1}{3}-\frac{p}{q}\right|>\frac{\log\log q}{9q^2\log q}.$$

§ 2. **Lemma. Lemma.** *Under the same assumptions as in Theorem,*

$$\left|\tanh\frac{1}{k}-\frac{p}{q}\right|>\frac{\log\log q}{\gamma_N q^2\log q}$$