35. *A Divisor Problem. I*

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(Communicated by Shokichi IYANAGA, M. J. A., June 8, 1993)

§1. Introduction. Let α be a real number ≥ 1. For any integer $n \geq 1$, let $\tau_{\alpha}(n)$ be the number of the divisors d of n of the form $d = [\alpha m]$, where m is an integer and $[\cdot]$ is the Gauss symbol. We are concerned with the asymptotic behavior of the sum

$$\sum_{n \leq X} \tau_{\alpha}(n)$$

as $X \to \infty$.

When $\alpha = 1$, $\tau_{\alpha}(n)$ is the usual divisor function $\tau(n)$ and if we put

$$\sum_{n \leq X} \tau(n) = X \log X + (2\gamma - 1)X + \Delta(X)$$

with the Euler constant γ, then Dirichlet proved first that

$$\Delta(X) \ll \sqrt{X}$$

and Voronoi proved later that

$$\Delta(X) \ll X^{\frac{1}{3}}.$$

The refinement of these results, namely, Dirichlet's divisor problem, has been the subject of many mathematicians (cf. Chap. XII of Titchmarsh [6] and Iwaniec and Mozzochi [4], for example). In this article, we shall evaluate our sum for an irrational α. The sum for a rational α will be treated in the subsequent paper.

To state our result, let $\psi(x)$ be a non-decreasing positive function of $x \geq 1$. An irrational number α is said to be of type $< \psi$ if

$$q \| q\alpha \| \geq \frac{1}{\psi(q)} \text{ for all integer } q \geq 1,$$

where $\| x \| = \min(\{x\}, 1 - \{x\})$ and $\{x\}$ is the fractional part of x (cf. Kuipers and Niederreiter [5]). Now our result may be stated as follows.

Theorem. *Let α be an irrational number > 1 and $\frac{1}{\alpha}$ be of type $< \psi$. Then we have for $X > X_0$,*

$$\sum_{n \leq X} \tau_{\alpha}(n) = \frac{1}{\alpha}(X \log X + (2\gamma - 1)X) + X\left(\left\{\frac{1}{\alpha}\right\} - \sum_{n=1}^{\infty} \frac{\left\{\frac{n+1}{\alpha}\right\}}{n(n+1)}\right)$$
$$+ O(X^{\frac{2}{5}} \log (X\psi(X))).$$

Remark 1. To get the remainder term $O(\sqrt{X})$ for any α is simple if we estimate S_4 and S_5 below trivially. So to refine $O(\sqrt{X})$ up to the above remainder term will be the main part of this article.

Remark 2. It is more suggestive to write

$$X\left(\left\{\frac{1}{\alpha}\right\} - \sum_{n=1}^{\infty} \frac{\left\{\frac{n+1}{\alpha}\right\}}{n(n+1)}\right)$$