20. *Notes on the Ideal Class Groups of the p-Class Fields of Some Algebraic Number Fields*

By Katsuya MIYAKE
Department of Mathematics, College of General Education, Nagoya University

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1. In our recent work [7], we studied the structure of the ideal class group of the p-class fields of quadratic number fields. As we indicated there, our methods may be applicable to wider varieties of number fields than that of quadratic fields. Here we treat some cases of Abelian cubic fields and of relative quadratic extensions where we find a little more complicated structures than in quadratic number fields.

2. We fix an odd prime p. Let k be an algebraic number field of finite degree, $\tilde{k}$ the Hilbert p-class field of k and $\tilde{\tilde{k}}$ that of $\tilde{k}$, i.e. the second p-class field of k. We denote the p-primary parts of the ideal class groups of k and of $\tilde{k}$ by $\mathrm{Cl}^{(p)}(k)$ and by $\mathrm{Cl}^{(p)}(\tilde{k})$, respectively. We suppose that the p-rank of $\mathrm{Cl}^{(p)}(k)$ is larger than 1 because $\mathrm{Cl}^{(p)}(\tilde{k})$ would be trivial if otherwise.

For simplicity, we put $C:=\mathrm{Cl}^{(p)}(k)$ and $G:=\mathrm{Gal}(\tilde{\tilde{k}}/k)$ throughout this paper. Denote the alternative product of C by itself by $C\wedge C$, and the lower central series of G by

$$G_1=G\supset G_2=[G_1, G]\supset G_3=[G_2, G]\supset\cdots.$$

Then $C\wedge C$ may be identified with the Schur multiplier of C (cf. e.g. Karpilovsky [3], 2.6.7 Theorem). The quotient group G/G_3 is a central extension of $G/G_2=\mathrm{Gal}(\tilde{k}/k)$ by the kernel G_2/G_3 which lies in both of the commutator subgroup and the center of G/G_3. Since G/G_2 is isomorphic to C by the Artin map of class field theory, there is a canonical surjective homomorphism of $C\wedge C$ onto G_2/G_3 (cf. e.g. [4], Theorem 4).

Let φ be an automorphism of $G=\mathrm{Gal}(\tilde{\tilde{k}}/k)$ and $\langle\varphi\rangle$ the cyclic group generated by it. Then $\langle\varphi\rangle$ acts not only on the abelian groups $G_2=[G, G]$ and G/G_2 but also on C through the Artin map. Define an action of $\langle\varphi\rangle$ on $C\wedge C$ by $(a\wedge b)^{\varphi}:=a^{\varphi}\wedge b^{\varphi}$ for $a, b\in C$.

Proposition 1. *For an algebraic number field k of finite degree, there exists a surjective $\langle\varphi\rangle$-homomorphism of $C\wedge C$ onto G_2/G_3.*

Proof. For $\alpha, \beta\in G$, the commutator $[\alpha, \beta]$ mod G_3 depends only upon the cosets $\alpha\cdot G_2$ and $\beta\cdot G_2$; therefore, by assigning $[\alpha, \beta]$ mod G_3 to the pair of $\alpha\cdot G_2$ and $\beta\cdot G_2$, we have a well defined surjective homomorphism from the alternative product $(G/G_2)\wedge(G/G_2)$ onto G_2/G_3; since $[\alpha, \beta]^{\varphi}=[\alpha^{\varphi}, \beta^{\varphi}]$, the proposition is clear.

We shall need the following fact (cf. e.g. [5], §2, Proposition 4).