16. On the Ideal Class Groups of the p-Class Fields of Quadratic Number Fields

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1. We fix an odd prime p. Let k be a quadratic number field and $\tilde{k}$ the Hilbert p-class field of k. Denote the p-primary parts of the ideal class groups of k and of $\tilde{k}$ by $\mathrm{Cl}^{(p)}(k)$ and by $\mathrm{Cl}^{(p)}(\tilde{k})$, respectively.

If the p-rank of $\mathrm{Cl}^{(p)}(k)$ is less than or equal to one, $\mathrm{Cl}^{(p)}(\tilde{k})$ is trivial. In fact, $\mathrm{Gal}(\tilde{k}/k)$ is then cyclic, and does not have any essential central extensions because the Schur multiplier of it is trivial.

If the p-rank of $\mathrm{Cl}^{(p)}(k)$ is greater than one, however, $\mathrm{Cl}^{(p)}(\tilde{k})$ is not trivial anymore. We see by Nomura [4] that $\tilde{k}/k$ has a non-trivial unramified central extension; in fact, we can show the following theorem by mathematical induction with Theorem 1 of [4]:

Theorem 1. *Suppose that the p-rank r of $\mathrm{Cl}^{(p)}(k)$ of a quadratic number field k is greater than one. Then $\tilde{k}/k$ has an unramified central extension $K/\tilde{k}/k$ whose group $\mathrm{Gal}(K/k)$ is isomorphic to the metabelian group D,*

$$D=\langle a_i, c_{i,j} \mid i=1,\cdots,r,\ j=i+1,\cdots,r\rangle,\quad a_i^{\varepsilon(i)}=c_{i,j}^{\varepsilon(i)}=1,\quad [a_i,a_j]=c_{i,j},$$
$$[a_i,c_{m,n}]=[c_{i,j},c_{m,n}]=1,\quad i=1,\cdots,r,\quad j=i+1,\cdots,r,\quad 1\leq m<n\leq r,$$

where the abelian group $\mathrm{Cl}^{(p)}(k)$ is of type $(\varepsilon(1),\cdots,\varepsilon(r))$, $e(i)=p^{e_i}$, $i=1,\cdots,r$, $1\leq e_1\leq\cdots\leq e_r$. In particular, we have $|\mathrm{Cl}^{(p)}(\tilde{k})|\geq\prod_{i=1}^{r}\varepsilon(i)^{(r-i)}$ and p-rank $(\mathrm{Cl}^{(p)}(\tilde{k}))\geq\binom{r}{2}$.

For simplicity, put $C:=\mathrm{Cl}^{(p)}(k)$ and $G:=\mathrm{Gal}(\hat{k}/k)$ where $\hat{k}$ is the Hilbert p-clase field of $\tilde{k}$; denote the alternative product of C by itself by $C\wedge C$, and the lower central series of G by

$$G_1=G\supset G_2=[G_1,G]\supset G_3=[G_2,G]\supset\cdots.$$

Then $C\wedge C$ may be identified with the Schur multiplier of C, and is isomorphic to the commutator group

$$[D,D]=\langle c_{i,j} \mid 1\leq i<j\leq r\rangle$$

of D of the theorem. Since $[D,D]$ is contained in the center of D, we see

Corollary. *Let the notation and the assumptions be as above. Then the field K of the theorem is the maximal unramified central extension of $\tilde{k}/k$; hence, in particular, G/G_3 is isomorphic to the group D of the theorem, and G_2/G_3 is to $C\wedge C$.*

It is possible to give a better estimate of the size of $\mathrm{Cl}^{(p)}(\tilde{k})$ than that of Theorem 1 in case of an imaginary quadratic number field k; in fact, k