15. A Note on a Deformation of Dirichlet's Class Number Formula

By Shigeru KATAYAMA
College of Engineering, Tokushima Bunri University

(Communicated by Shokichi IYANAGA, M. J. A., March 12, 1992)

§ 0. **Introduction.** In [3], Prof. T. Ono obtained interesting results from a deformation of Dirichlet's class number formula for real quadratic fields $Q(\sqrt{p})$, where p is a prime number of the form $p=4N+1$. In [2], the author gave a similar deformation in the case where p is a prime number of the form $p=4N+3$.

After the completion of [2], the author found that Dirichlet had already given a deformation of the class number formula for binary quadratic forms ([1], § 107–§ 110 and § 138–§ 140), which is, however, somewhat complicated. The purpose of this note is to give a more simple formula for any real quadratic field using the same methods as [2] and [3]. To be more precise, let m be a square-free positive integer, ε the fundamental unit >1 of the real quadratic field $Q(\sqrt{m})$ and h the class number of $Q(\sqrt{m})$. We denote by d the discriminant of $Q(\sqrt{m})$. The discriminant d is written in the form $d=P\,(\equiv 1 \bmod 4)$, $4P$ or $8P$, where $P=1$ or $P=p_1p_2\cdots p_r$ ($p_1, p_2, \cdots, p_r$ are distinct odd prime numbers). ζ denotes a primitive dth root of unity. Let χ be a Kronecker character belonging to $Q(\sqrt{m})$, and $L(s,\chi)$ the corresponding L-series. As usual, we denote by ϕ the Euler function, and by μ the Möbius function. For the sake of simplicity, we denote $\phi(d)/4$ by v. For any integer $1\leqq t\leqq v$, define τ_t by putting

$$\tau_t=((\phi(d)/\phi(d/n))\cdot\mu(d/n)-\chi(t)\sqrt{d})/2, \qquad \text{where } n=(t,d).$$

We also define W as follows

$$W=\begin{cases}0, & \text{if } d \text{ has at least two distinct prime factors,}\\ 1, & \text{otherwise.}\end{cases}$$

Then our main theorem reads.

Theorem. *With the above notations, we have*

$$\sqrt{m^W}\,\varepsilon^h=2\sum_{j=0}^{v-1}d_j+d_v.$$

Here d_j are determined by the following recurrence relation.

$$jd_j=\sum_{t=1}^{j}\alpha_t d_{j-t} \quad (d_0=1,\ 1\leqq j\leqq v),$$

where $\alpha_t=-\tau_t$.

§ 1. **Dirichlet's formula.** It is known that (cf. [4])

$$h\kappa=L(1,\chi), \quad \text{where } \kappa=\frac{\log\varepsilon^2}{\sqrt{d}} \quad \text{and} \quad L(1,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n}$$

and

$$\sum_{r \bmod d}\chi(r)\zeta^{nr}=\chi(n)\sqrt{d} \qquad \text{(the Gauss sum).}$$