14. Global Solvability and Hypoellipticity on the Torus for a Class of Differential Operators with Variable Coefficients

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1. Notations and results. The purpose of the present note is to give conditions for global solvability and hypoellipticity for a class of second order differential operators on the two dimensional torus $\boldsymbol{T}^2$. The principal result is a necessary and sufficient condition for the global solvability in terms of small denominator type estimates, a phenomena known so far only for differential operators with constant coefficients. We recall the well known result of S. Greenfield and N. Wallach [5] showing that the operator D_x+cD_y, $c\in\boldsymbol{R}\backslash 0$ is globally hypoelliptic on $\boldsymbol{T}^2$ if and only if c is an irrational non Liouville number, despite that it is always locally non-hypoelliptic. J. Hounie [6] proved a necessary and sufficient condition for global solvability for first order systems $\partial_t+b(t)A$, where A is an essentially self-adjoint operator, while D. Fujiwara and H. Omori [2] established global hypoellipticity for $D_x^2+\varphi(x)D_y^2$, φ being $C^\infty(\boldsymbol{R})$ 2π periodic real-valued function, identically equal to 0 and 1 on some subintervals of $[0, 2\pi]$. Recently, the third author studied global hypoellipticity of a Mathieu operator on $\boldsymbol{T}^2$.

The present paper examines second order differential operators on the two dimensional torus $\boldsymbol{T}^2=\boldsymbol{R}^2/2\pi\boldsymbol{Z}^2$ of the following type

$$P=(D_x+ia(x)D_y)(D_x+ib(x)D_y)+\gamma(x)D_y+c(x), \tag{1}$$

where $\gamma(x)$ equals either 0 or $(a'(x)-b'(x))$, $D_z=i^{-1}\partial_z$, $z=x$ or y, and where $a(x)$, $b(x)$, $c(x)\in C^\infty(\boldsymbol{T})$, i.e., 2π periodic C^∞ complex-valued functions on $\boldsymbol{R}$. We will study the following equation in the space of periodic distributions $\mathcal{D}'(\boldsymbol{T}^2)$

$$Pu=f, \qquad f\in\mathcal{D}'(\boldsymbol{T}^2). \tag{2}$$

When $c\equiv 0$, the operator P has a remarkable property that the solutions to the homogeneous equation $Pu=0$ on $\boldsymbol{T}^2$ are written explicitly.

We say that P is globally solvable (resp. hypoelliptic) if for every $f\in C^\infty(\boldsymbol{T}^2)$ there exists a $u\in\mathcal{D}'(\boldsymbol{T}^2)$ satisfying (2) (resp. $u\in C^\infty(\boldsymbol{T}^2)$ when $Pu\in C^\infty(\boldsymbol{T}^2)$ and $u\in D'(\boldsymbol{T}^2)$). Similarly, P is said to be globally hypoelliptic in the Gevrey class $G^s(\boldsymbol{T}^2)$ if $Pu\in G^s(\boldsymbol{T}^2)$ and $u\in G^{s'}(\boldsymbol{T}^2)$ implies $u\in G^s(\boldsymbol{T}^2)$. P is said to be locally solvable (resp. hypoelliptic) at a point p if there

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