# 12. *Kostant's Theorem for a Certain Class of Generalized Kac-Moody Algebras*

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**Introduction.** Let $A=(a_{ij})_{i,j\in I}$ be a real $n\times n$ matrix satisfying the following conditions:

(C1) either $a_{ii}=2$ or $a_{ii}\leq 0$;

(C2) $a_{ij}\leq 0$ if $i\neq j$, and $a_{ij}\in \boldsymbol{Z}$ if $a_{ii}=2$;

(C3) $a_{ij}=0$ implies $a_{ji}=0$.

Such a matrix is called a *generalized GCM* (=GGCM). And let $\mathfrak{g}(A)$ be the generalized Kac-Moody algebra (=GKM algebra), over the complex number field $\boldsymbol{C}$, associated to the above GGCM $A$. Then, we have the root space decomposition: $\mathfrak{g}(A)=\mathfrak{h}\oplus\sum_{\alpha\in\Delta}^{\oplus}\mathfrak{g}_{\alpha}$, where $\mathfrak{h}$ is the Cartan subalgebra, and $\Delta$ the root system of $(\mathfrak{g}(A),\mathfrak{h})$. Let $J$ be a subset of $I^{re}:=\{i\in I\,|\,a_{ii}=2\}$. And put $\mathfrak{n}_J^{\pm}:=\sum_{\alpha\in\Delta_J^+}^{\oplus}\mathfrak{g}_{\pm\alpha}$, $\mathfrak{u}^{\pm}:=\sum_{\alpha\in\Delta^+(J)}^{\oplus}\mathfrak{g}_{\pm\alpha}$, $\mathfrak{m}:=\mathfrak{n}_J^-\oplus\mathfrak{h}\oplus\mathfrak{n}_J^+$, where $\Delta_J^+:=\Delta\cap\sum_{i\in J}\boldsymbol{Z}_{\geq 0}\alpha_i$, $\Delta^+(J):=\Delta^+\backslash\Delta_J^+$. In this paper, we study the homology $H_j(\mathfrak{u}^-,L(\Lambda))$ of $\mathfrak{u}^-$ and the cohomology $H^j(\mathfrak{u}^+,L(\Lambda))$ of $\mathfrak{u}^+$ with coefficients in the irreducible highest weight $\mathfrak{g}(A)$-module $L(\Lambda)$ with highest weight $\Lambda\in\mathfrak{h}^*$. And we prove "Kostant's homology and cohomology theorem" for symmetrizable GKM algebras associated to GGCMs satisfying the following condition ($\hat{\text{C}}$1) instead of (C1) above:

($\hat{\text{C}}$1) either $a_{ii}=2$ or $a_{ii}=0$.

This result is a generalization of Kostant's Theorem for Kac-Moody algebras, which was proved by J. Lepowsky and H. Garland ([2] and [5]), or the classical result of B. Kostant himself [4] for finite dimensional complex semi-simple Lie algebras.

**§1. Preliminaries for GKM algebras.** We prepare some basic results for GKM algebras which will be needed later. For details, see [1] and [3]. Let $\mathfrak{g}(A)$ be the GKM algebra associated to a GGCM $A$, with the Cartan subalgebra $\mathfrak{h}$, simple roots $\Pi=\{\alpha_i\}_{i\in I}$, and simple co-roots $\Pi^{\vee}=\{\alpha_i^{\vee}\}_{i\in I}$. From now on, we always assume that the GGCM $A=(a_{ij})_{i,j\in I}$ is symmetrizable, and that $J$ is a subset of $I^{re}=\{i\in I\,|\,a_{ii}=2\}$. We call an $\mathfrak{h}$-module $V$ *$\mathfrak{h}$-diagonalizable* if $V$ admits a weight space decomposition: $V=\sum_{\lambda\in\mathcal{P}(V)}^{\oplus}V_{\lambda}$, where $\mathcal{P}(V)$ is the set of all weights of $V$.

**Definition** ([6]). $\mathcal{O}_J$ is the category of all $\mathfrak{m}$-modules whose objects $V$ satisfy the following:

(1) $V$ is $\mathfrak{h}$-diagonalizable;

(2) the weight space $V_{\mu}$ is finite dimensional for all $\mu\in\mathcal{P}(V)$;

(3) there exist a finite number of elements $\lambda_i$ $(1\leq i\leq s)$ in $\mathfrak{h}^*:=$