83. A Note on Certain Infinite Products

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1. Statement of result. Let M be a positive integer, χ a real non-principal primitive character modulo M, $L(s, \chi)$ the associated L-series and $\zeta_M = \exp(2\pi i/M)$. Given a sequence $a(1), a(2), a(3), \cdots$ of integers such that $a(n) = O(n^c)$ for some $c > 0$, we define, for $Im(z) > 0$,

$$f_\chi(z) = \exp(2\pi iaz) \prod_{h=0}^{M-1} \prod_{n=1}^{\infty} (1 - \zeta_M^h q(\lambda)^n)^{\chi(h)a(n)}, \tag{1}$$

where $q(\lambda) = \exp(2\pi iz/\lambda)$, $\lambda > 0$ and a is a real number. Then the infinite product converges absolutely and uniformly in every compact subset of the upper half plane H. Hence $f_\chi(z)$ is holomorphic in H. To state our theorem, let $\phi(s)$ be a convergent Dirichlet series defined by

$$\phi(s) = \sum_{n=1}^{\infty} a(n)n^{-s}.$$

Theorem. *Assume that $\phi(s)$ can be continued through the whole s-plane as a non-zero meromorphic function with a finite number of poles and that there exists a real number k such that*

$$f_\chi(-1/z) = (z/i)^k f_\chi(z). \tag{2}$$

Then $(\lambda/M)^2$ is an integer, $a = k = 0$ and $f_\chi(z)$ is given by

$$f_\chi(z) = \prod_{m|(\lambda/M)^2} \varphi_\chi(mz)^{c(m)}, \tag{3}$$

where

$$\varphi_\chi(z) = \prod_{h=0}^{M-1} \prod_{n=1}^{\infty} (1 - \zeta_M^h q(\lambda)^n)^{\chi(h)\chi(n)},$$

and $c(m)$, defined for m dividing $(\lambda/M)^2$, are integers such that $c(m) = \chi(-1)c((\lambda/M)^2/m)$ for any divisor m of $(\lambda/M)^2$.

Conversely, let $(\lambda/M)^2$ be an integer and let $c(m)$, for integers m dividing $(\lambda/M)^2$, be arbitrary integers such that $c(m) = \chi(-1)c((\lambda/M)^2/m)$ for any divisor m of $(\lambda/M)^2$. Further, define $f_\chi(z)$ by (3). Then $f_\chi(z)$ satisfies $f_\chi(-1/z) = f_\chi(z)$.

Remark. In case $\lambda = M$, $\varphi_\chi(z)$ coincides with $\eta_3(\chi; z)$ which was first defined in Katayama [1].

2. Lemmas. For any $y > 0$, we put

$$G(y) = -\{\log f_\chi(iy) + 2a\pi y\}.$$

Then from (1), we have

$$G(y) = T(\chi) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(m)a(n)}{m} \exp(-2mn\pi y/\lambda), \tag{4}$$

where $T(\chi)$ is the Gaussian sum defined by

$$T(\chi) = \sum_{h=0}^{M-1} \chi(h)\zeta_M^h.$$