# 82. On the Regularity of Prehomogeneous Vector Spaces

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**Introduction. 0.1.** Let $G$ be a complex linear algebraic group, and $V(\neq 0)$ a complex vector space on which $G$ acts via a rational representation. Let $V^\vee$ denote the dual space of $V$, on which $G$ acts naturally.

**0.2.** If $V$ has an open $G$-orbit, then $(G, V)$ is called a *prehomogeneous vector space*. A prehomogeneous vector space $(G, V)$ is called *regular* if there exists a (non-zero) relatively invariant polynomial function $f(x) = f(x_1, \cdots, x_n)$ on $V$ such that

$$\det\left(\frac{\partial^2 \log f}{\partial x_i \partial x_j}\right) \not\equiv 0. \tag{0.3}$$

Let $(G, V)$ be a regular prehomogeneous vector space, and take $f$ so that (0.3) holds. Then the following assertions hold [5, §2].

(0.4) $(G, V^\vee)$ is also prehomogeneous.

(Moreover $(G, V^\vee)$. is regular.)

(0.5) Let $O$ (resp. $O^\vee$) be the open $G$-orbit in $V$ (resp. $V^\vee$). Then $O \simeq O^\vee$.

(0.6) Let $V \setminus O = (\cup_{i=1}^{l} S_i) \cup (\cup_{j=1}^{m} T_j)$ (resp. $V^\vee \setminus O^\vee = (\cup_{i=1}^{l'} S_i^\vee) \cup (\cup_{j=1}^{m'} T_j^\vee)$) be the irreducible decomposition, where the codimension of $S_i$ and $S_i^\vee$ (resp. $T_j$ and $T_j^\vee$) are one (resp. greater than one). Then $l = l'$.

Continue to assume $(G, V)$ regular and prehomogeneous. Bearing (0.5) in mind, H. Yoshida posed the following problem.

**Problem 1.** $V \setminus O \simeq V^\vee \setminus O^\vee$?

Bearing (0.6) in mind, let us also consider the following problem.

**Problem 2.** *For any integer* $c > 1$, card $\{j \mid \operatorname{codim} T_j = c\} =$ card $\{j \mid \operatorname{codim} T_j^\vee = c\}$?

We shall settle Problem 1 negatively in §1, and Problem 2 affirmatively in §2.

**Convention.** If no explanation is given, a lowercase letter should be understood as an element of the set denoted by the same uppercase letter. We define the Bruhat order of a Coxeter group so that the identity element is minimal. We denote the complex number field by $\boldsymbol{C}$, and the rational integer ring by $\boldsymbol{Z}$.

**§1. 1.1.** The following argument will be used. Let $Z$ be a complex algebraic variety. We may assume that $Z$ is defined over a field which is finitely generated over the rational number field. Then we can consider its specialization at enough general primes. Especially we obtain a variety over a finite field whenever the characteristic $p$ and the cardinality $q$ of the field are large enough. Let $|Z| = |Z|(q)$ be the cardinality of the rational points of the variety obtained above. We understand that $|Z|$ is a function of $q$,