79. *On the Starlikeness of the Alexander Integral Operator*

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Abstract : Denote by A the class of functions $f(z)$ analytic in the unit disk D and normalised so that $f(0)=f'(0)-1=0$. For $f(z)\in A$, let $F(z)=\int_0^z [f(t)/t]\,dt$ for $z\in D$. We find estimate on β so that $\mathrm{Re}\, f'(z)>-\beta$ will ensure the starlikeness of $F(z)$. Our conclusion improves the well-known results.

1. Introduction. Denote by A the class of functions $f(z)$ which are analytic in the unit disc $D=\{z:|z|<1\}$ and normalised so that $f(0)=f'(0)-1=0$. Let R_α be the subclass of A satisfying $\mathrm{Re}\, f'(z)>\alpha$ for $z\in D$ and S^* be the subset of starlike functions, i. e.

$$S^*=\{f(z)\in A:\mathrm{Re}[zf'(z)/f(z)]>0 \text{ for } z\in D\}.$$

For $f(z)\in A$, let

$$F(z)=\int_0^z [f(t)/t]\,dt \quad z\in D. \tag{1}$$

This integral operator was first introduced by J. W. Alexander. In paper [1], R. Singh and S. Singh showed that if $f(z)\in R_0$, then $\mathrm{Re}[F(z)/z]>1/2$ $(z\in D)$, and if $\mathrm{Re}\, f'(z)>-1/4$, then $F(z)\in S^*$. Recently M. Nunokawa and D. K. Thomas [2] improved the second result by showing that if $\mathrm{Re}\, f'(z)>-0.262$, then $F(z)\in S^*$.

In this paper we will improve both two conclusions.

2. Results and proofs. In proving our results, we need the following lemmas.

Lemma 1 ([3]) . *Let $f(z)$ be analytic and $g(z)$ convex in D (that is, in D, $g(z)$ satisfies $\mathrm{Re}[1+zg''(z)/g'(z)]>0$). If $f(z)\prec g(z)$ $(z\in D)$, then we have*

$$z^{-1}\int_0^z f(t)\,dt\prec z^{-1}\int_0^z g(t)\,dt,$$

where "$\prec$" denotes the subordination.

Lemma 2 ([4]) . *If $g(z)\in K$ — the normalised class of convex functions, then*

$$G(z)=\frac{2}{z}\int_0^z g(t)\,dt\in K.$$

Lemma 3 ([5]). *Let $w(z)$ be a non-constant regular function in D, $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at z_0, then we have $z_0w'(z_0)=kw(z_0)$, where k is a real number,*