76. Criteria for the Finiteness of Restriction of $U(\mathfrak{g})$-modules to Subalgebras and Applications to Harish-Chandra Modules

By Hiroshi YAMASHITA

Department of Mathematics, Kyoto University

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Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. In this paper, we give simple and useful criteria for finitely generated $U(\mathfrak{g})$-modules H to remain finite under the restriction to subalgebras $A \subset U(\mathfrak{g})$, by using the algebraic varieties in $\mathfrak{g}^*$ associated to H and A. It is shown that, besides the finiteness, the $U(\mathfrak{g})$-modules H satisfying our criteria preserve some important invariants under the restriction.

Applying the criteria to Harish-Chandra modules of a semisimple Lie algebra $\mathfrak{g}$, we specify among other things, a large class of Lie subalgebras of $\mathfrak{g}$ on which all the Harish-Chandra modules are of finite type. This allows us to extend largely the finite multiplicity theorems for induced representations of a semisimple Lie group, established in our earlier work [8].

1. Associated varieties for finitely generated $U(\mathfrak{g})$-modules. We begin with defining three important invariants: the associated variety, the Bernstein degree and the Gelfand-Kirillov dimension, of finitely generated modules over a complex Lie algebra (cf. [6]).

Let V be a finite-dimensional complex vector space. We denote by $S(V) = \bigoplus_{k=0}^{\infty} S^k(V)$ the symmetric algebra of V, where $S^k(V)$ is the homogeneous component of $S(V)$ of degree k. Let $M = \bigoplus_{k=0}^{\infty} M_k$ be a finitely generated, nonzero, graded $S(V)$-module, on which $S(V)$ acts in such a way as $S^k(V)\, M_{k'} \subset M_{k+k'}$ $(k, k' \geq 0)$. Then each homogeneous component M_k of M is finite-dimensional.

Proposition 1 (Hilbert-Serre, see [9, Ch. VII, §12]). (1) *There exists a unique polynomial* $\varphi_M(q)$ *in* q *such that* $\varphi_M(q) = \dim(M_0 + M_1 + \cdots + M_q)$ *for sufficiently large* q.

(2) *Let* $(c(M)/d(M)!)q^{d(M)}$ *be the leading term of* φ_M. *Then* $c(M)$ *is a positive integer, and the degree* $d(M)$ *of this polynomial coincides with the dimension of the associated algebraic cone*

(1.1) $\nu(M) := \{\lambda \in V^* \mid f(\lambda) = 0 \text{ for all } f \in \mathrm{Ann}_{S(V)} M\}.$

Here, $\mathrm{Ann}_{S(V)} M$ *denotes the annihilator of* M *in* $S(V)$, V^* *the dual space of* V, *and* $S(V)$ *is identified with the polynomial ring over* V^* *in the canonical way.*

For a finite-demensional complex Lie algebra $\mathfrak{g}$, let $(U_k(\mathfrak{g}))_{k=0,1\ldots}$ denote the natural filtration of enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, where $U_k(\mathfrak{g})$ is the subspace of $U(\mathfrak{g})$ generated by elements $X_1\ldots X_m$ with $m \leq k$ and $X_j \in \mathfrak{g}$ $(1 \leq j \leq m)$. We identify the associated commutative ring $\mathrm{gr}\, U(\mathfrak{g}) = \bigoplus_{k \geq 0}$