74. *A Criterion for Algebraicity of Analytic Set Germs*

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In this paper we characterize the germs of algebraic subsets among the germs of analytic subsets by validity of an inequality between orders and degrees for polynomial functions on them.

Throughout this paper K denotes the field $\boldsymbol{C}$ or $\boldsymbol{R}$. Let S be a germ at 0 of an analytic subset of an open neighborhood of $0 \in K^n$ and $\nu_{S,0}(f)$ the vanishing order of $f \in K[x_1,\dots,x_n]$ at 0 along S. To be accurate, if $\mathfrak{i}_0 \subset K\{x_1,\dots,x_n\}$ is the analytic ideal of S at 0, $\mathfrak{m} \equiv (x_1,\dots,x_n) \subset K\{x_1,\dots,x_n\}$ the maximal ideal at 0 and if $f_0 \in K\{x_1,\dots,x_n\}$ is the germ of f at 0, we put

$$\nu_{S,0}(f) = \max\{r \in \boldsymbol{N} : f_0 \in \mathfrak{m}^r + \mathfrak{i}_0\}.$$

Theorem. *Let S be a germ at 0 of an analytic subset of an open neighborhood of $0 \in K^n$. Suppose that S is irreducible and of positive dimension. Then the following conditions are equivalent.*

($*$) *S is an analytic irreducible component of the germ of an algebraic subset.*

($**$) *There exists $a \in \boldsymbol{R}$ such that $a \cdot \deg f \geqq \nu_{S,0}(f)$ for any $f \in K[x_1,\dots,x_n]$ that does not vanish identically on S. Such an a must satisfy $a \geqq 1$.*

We may replace $\nu_{S,0}(f)$ in the above by the reduced order $\bar{\nu}_{S,0}(f) \equiv \lim_{k\to\infty} \nu_{S,0}(f^k)/k$ (cf. [3]). Our theorem exhibits an analogy to Sadullaev's theorem [4] which characterizes the algebraic subsets by a growth estimate of polynomial functions (cf. [1] for "analogy").

The complex case of $(*) \Rightarrow (**)$ is already known (a slight modification of [1], Thm. A*, (2.1), where the author has carelessly omitted the non-vanishing condition for f). The real case of $(*) \Rightarrow (**)$ easily follows from the complex case. Since $\nu_{S,0}(f) \geqq \deg f$ holds for homogeneous f which does not vanish identically on S, $a \geqq 1$ follows. Thus we have only to prove $(**) \Rightarrow (*)$.

Proof of $(**) \Rightarrow (*)$. Let $J \subset K[x_1,\dots,x_n]$ be the ideal of all polynomials which vanish on S. J defines the minimal algebraic subset $\tilde{T} \subset K^n$ such that its germ T at 0 includes S. Let p and q denote the dimensions of S and T respectively. In complex case it is well-known that $\dim \tilde{T} = q$. In real case, the same equality follows from the following:

$$\dim_{\boldsymbol{R}} T = \dim_{\boldsymbol{C}} T^{\boldsymbol{C}} = \dim_{\boldsymbol{C}} (T^{\boldsymbol{C}})^{\sim} \geqq \dim_{\boldsymbol{R}} (T^{\boldsymbol{C}})^{\sim} \cap \boldsymbol{R}^n \geqq \dim_{\boldsymbol{R}} \tilde{T} \geqq \dim_{\boldsymbol{R}} T,$$

where $T^{\boldsymbol{C}}$ denotes the complexification of T and the first equality follows from [2], V, Prop.3. Let us put

$$A \equiv K[x_1,\dots,x_n],\ A_k \equiv \{f \in A : \deg f \leqq k\},\ J_k \equiv J \cap A_k.$$

A_k/J_k is the set of the germs of polynomial functions on $\tilde{T}$ of degree $\leqq k$. We can naturally identify K^n with an affine chart of the projective space $\boldsymbol{KP}^n$. Then, by the theory of Hibert polynomial applied to the closure of