5. *On Fundamental Units of Real Quadratic Fields with Norm +1*

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1. In our previous paper [2], we gave a new explicit form of the fundamental units of real quadratic fields with norm -1. In this note, we shall show that similar results also hold for the fundamental units of real quadratic fields with norm $+1$.

Let m be a positive integer which is not a perfect square and K be the real quadratic field $\boldsymbol{Q}(\sqrt{m})$. ε_0 denotes the fundamental unit of K. N denotes the norm map from K to $\boldsymbol{Q}$, and for any $x \in K$, $\bar{x}$ will denote the conjugate of x. We put

$R_+ = \{K$: *real quadratic fields with* $N\,\varepsilon_0 = +1\}$,

$E_+ = \{\varepsilon$: *units of real quadratic fields such that* $N\varepsilon = +1$ *and* $\varepsilon + \bar{\varepsilon} \geq 3\}$.

Then it is easy to see $R_+ \subset \{\boldsymbol{Q}(\sqrt{a^2+4a}) : a \in \boldsymbol{N}\}$, where $\boldsymbol{N}$ is the set of all the natural numbers.

Fix now a unit $\varepsilon = (t+2+u\sqrt{m})/2 = (t+2+\sqrt{t(t+4)})/2 \in E_+$ $(t>0)$ for a while, and denote $\varepsilon^n = (t_n+2+u_n\sqrt{m})/2$.

Since $t_n+2 = \varepsilon^n + \bar{\varepsilon}^n$, we have

$$t_{n+1} = \varepsilon^{n+1} + \bar{\varepsilon}^{n+1} - 2 = (\varepsilon+\bar{\varepsilon})(\varepsilon^n+\bar{\varepsilon}^n) - \varepsilon^{n-1} - \bar{\varepsilon}^{n-1} - 2$$
$$= (t+2)(t_n+2) - (t_{n-1}+2) - 2 = (t+2)t_n - t_{n-1} + 2t \quad (n \geq 2).$$

Using the fact $t_1 = t$ and $t_2 = t^2+4t$ and this recurrence, we get inductively $t \mid t_n$ and $t_{n+1} - t_n = (t+1)t_n - t_{n-1} + 2t > (t+1)(t_n - t_{n-1})$ $(n \geq 2)$. Hence $t_{n+1} - t_n \geq t(t+3)(t+1)^{n-1}$ $(n \geq 1)$. Furthermore we have

$$(t_{n+1}-t_n)^2 = \{(\varepsilon^{n+1}+\bar{\varepsilon}^{n+1}) - (\varepsilon^n+\bar{\varepsilon}^n)\}^2 = (\varepsilon^{2n+2}+\bar{\varepsilon}^{2n+2}) + (\varepsilon^{2n}+\bar{\varepsilon}^{2n}) - 2(\varepsilon^{2n+1}+\bar{\varepsilon}^{2n+1})$$
$$-2(\varepsilon+\bar{\varepsilon}) + 4 = t_{2n+2} + 2 + t_{2n} + 2 - 2(t_{2n+1}+2) - 2(t+2) + 4 = tt_{2n+1}.$$

Therefore we have obtained the following lemma.

Lemma 1. *With the above notation, we have*

(i) $t_1 = t$, $t_2 = t^2+4t$, *and* $t_{n+1} = (t+2)t_n - t_{n-1} + 2t$ $(n \geq 2)$,

(ii) $t \mid t_n$ *and* $t_{n+1} - t_n \geq t(t+3)(t+1)^{n-1}$ $(n \geq 1)$,

(iii) $tt_{2n+1} = (t_{n+1} - t_n)^2$ $(n \geq 1)$.

Until now ε has been fixed. Now let ε vary in E_+ and write $t_n(\varepsilon) = \varepsilon^n + \bar{\varepsilon}^n - 2$.

Lemma 2. *For any* $\varepsilon \in E_+$ *and* $n \geq 2$, $t_n(\varepsilon)$ *is not a prime except in the case* $n=2$ *and* $\varepsilon = (3+\sqrt{5})/2$.

Proof. Suppose n decomposes into $n = ij$, where $i, j \geq 2$. Then, from (ii) of Lemma 1, $\varepsilon^n = (\varepsilon^i)^j$ implies $t_i(\varepsilon) \mid t_n(\varepsilon)$, and furthermore $t_i(\varepsilon) \geq t_2(\varepsilon) \geq 5t$, and $t_n(\varepsilon) \geq 5t_i(\varepsilon)$. Hence $t_n(\varepsilon)$ is not prime in this case.

Next, suppose $n \geq 2$ and $t(\varepsilon) = t \geq 2$. Then one gets $t(\varepsilon) \mid t_n(\varepsilon)$ and $t_n(\varepsilon) \geq$