78. A Note on the Class-number of Real Quadratic Fields with Prime Discriminants

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Introduction. In recent papers [6], [7], [8], we defined some new integer-valued p-invariants for any rational prime p congruent to 1 mod 4 and studied relationships among them. In particular, we defined in [6] the new p-invariant n_p by

$$|t_p/u_p^2-n_p|<1/2$$

through the fundamental unit

$$\varepsilon_p=(t_p+u_p\sqrt{p})/2 \quad (>1)$$

of real quadratic field $Q(\sqrt{p})$ with prime discriminant, which turned out to be very useful as far as $n_p\neq 0$ (i.e. $2t_p>u_p^2$).

In this paper, we shall introduce some more new p-invariants q_p, r_p, r_p^*, a_p, b_p and provide lower bounds for the class-number h_p of $Q(\sqrt{p})$ (Theorems 1, 2). Moreover, we shall show that if $Q(\sqrt{p})$ is of R-D type and $h_p=1$, 3 or 5, then n_p has certain simple multiplicative structures (Theorem 3).

§ 1. We first prove the following theorem which is fundamental throughout this paper, providing a lower bound for the class-number h_p of real quadratic field $Q(\sqrt{p})$ with prime discriminant.

Theorem 1. *For any prime p congruent to* 1 mod 4, *we denote by q_p the least prime number which splits completely in $Q(\sqrt{p})$, i.e. $(p/q_p)=1$, where $(\ /\ )$ means Legendre's symbol.*

Then if $n_p\neq 0$, $h_p\geqq \log n_p/\log q_p$ holds.

Proof. In the case $q_p\neq 2$, we proved this already in [6]. In the case $q_p=2$, we can prove the following lemma in a similar way as in Lemma 2 in [6]:

Lemma. *For any square-free positive integer D congruent to* 1 mod 8, *we denote by e the order of prime factors of* 2 *in the ideal class group of $Q(\sqrt{D})$.*

Then, the diophantine equation $x^2-Dy^2=\pm 4\cdot 2^e$ has at least one non-trivial solution, while for any integer e' such that $1\leqq e'<e$ the diophantine equation $x^2-Dy^2=\pm 4\cdot 2^{e'}$ has no non-trivial integral solution.

By using this lemma together with Lemma 1 in [6], in a similar way as in the proof of Theorem in [6] we can prove

$$q_p=2 \quad \text{and} \quad h_p\geqq \log n_p/\log 2$$

for any prime $p\equiv 1 \bmod 8$.

We next provide a lower bound r_p for the class-number of $Q(\sqrt{p})$