58. *Itô's Formula for Pseudo-processes*

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In [5] there have been introduced the so-called "pseudo-processes" and a stochastic calculus has been developed (in a certain direction). In [7], starting with an equivalent definition for pseudo-processes, a stochastic integral has been introduced and afterwards it has been developed a similar calculus but in other directions. What was missing in [7] is an Itô formula for the stochastic integral, which will be discussed in this article.

Let W be a nonvoid set. We shall denote by Fin $[0,1]$ the set of all finite subsets for the real interval $[0,1]$. We suppose that for every $I \in$ Fin $[0,1]$ there is given $\mathcal{F}_I=a$ tribe of parts from W such that $(\mathcal{F}_I)_{I\in \mathrm{Fin}[0,1]}$ is an increasing family. We denote by $\mathcal{A}_t=\{\cup \mathcal{F}_I\,;\, I \in \mathrm{Fin}\,[0,t]\}$ and $\mathcal{F}_t=$ the tribe generated by $\mathcal{A}_t$. It is considered to be given $P: \mathcal{A}_1 \to \boldsymbol{R}_+$ additive such that $P^I :=P|_{\mathcal{F}_I}$ is a probability measure for every $I \in \mathrm{Fin}\,[0,1]$. Let $t \in [0,1]$ be fixed; by $L^1(W, \mathcal{A}_t, P)$ we understand the set of all $H: \mathcal{A}_1 \to \boldsymbol{R}$ additive such that, for every $I \in \mathrm{Fin}\,[0,1]$, $H^I=H|_{\mathcal{F}_I}$ is a real measure, H^I is absolutely continuous with respect to P and the Radon-Nikodym derivative $\dfrac{dH^I}{dP}$ is $\mathcal{F}_t$-measurable a P^I integrable. When the pseudoprocess H_t has (locally) bounded densities and X is a usual martingale (or a semimartingale), then the stochastic integral $(H\otimes X)_t^I$ is given by the formula $\int_0^t \frac{dH_s^I}{dP}dM_s+\int_0^t \frac{dH_s^I}{dP}dA_s$ where the first is an Itô-type integral (with respect to a local martingale), the second is Stieltjes and $X=M+A$. For example (see [7]), if H_t is a pseudo-martingale, then $H\otimes X$ is well-defined and the result is an $\mathcal{F}_I \cap \mathcal{F}_t$-usual martingale. Hence, if $f \in C^2(R)$ then, by Itô, $f(H\otimes X)$ is a semimartingale.

In the sequel, the stochastic integral is made with respect to usual Brownian motion $B_t (0\le t\le 1)$ and the integrand admits bounded densities or $\int\left(\frac{dH_s^I}{dP}\right)^2 ds<+\infty$. Both imply that the stochastic integral is will defined. We have the following Itô formula

$$f((H\otimes B)_t^I)=f((H\otimes B)_0^I+(f'((H\otimes B)_-)\cdot H\otimes B)_t^I$$
$$+\frac{1}{2}\int_0^t f''(H\otimes B)_{s-}^I\left(\frac{dH_s^I}{dP}\right)^2 ds.$$

First remark that the integrals are well-defined: the densities of the first integral are bounded and for the second, we can use Schwarz inequality. Secondly, for the proof, it is sufficient to prove the formula for polynomials,