# 30. On Regular Subalgebras of a Symmetrizable Kac-Moody Algebra

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Let $\mathfrak{g}(A)$ be a Kac-Moody algebra with $A$ a symmetrizable generalized Cartan matrix (= GCM) over the complex number field $\boldsymbol{C}$. In this paper, we study its certain subalgebras called *regular subalgebras*. These subalgebras are defined as a natural infinite dimensional analogue of *regular semi-simple subalgebras* of a finite dimensional complex semi-simple Lie algebra in the sense of Dynkin. The latter plays an important role in the classification of semi-simple subalgebras (cf. [1]).

**§ 1. Definition of regular subalgebras.** Let $A$ be an $n\times n$ symmetrizable GCM, and $\mathfrak{h}$ be a Cartan subalgebra of the Kac-Moody algebra $\mathfrak{g}(A)$. Then we have the root space decomposition of $\mathfrak{g}(A)$:

$$\mathfrak{g}(A)=\mathfrak{h}\oplus\textstyle\sum_{\alpha\in\Delta}^{\oplus}\mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha=\{x\in\mathfrak{g}(A)\,;\,[h,x]=\langle\alpha,h\rangle x,\text{ for all }h\in\mathfrak{h}\}$ for $\alpha\in\mathfrak{h}^*$ (the algebraic dual of $\mathfrak{h}$), and $\Delta\subset\mathfrak{h}^*$ is the root system of $\mathfrak{g}(A)$ (see [3] for details). To define a *regular subalgebra* of $\mathfrak{g}(A)$, we introduce the notion of *fundamental* subset of $\Delta$.

**Definition 1.1.** A subset $\bar{\Pi}=\{\beta_1,\cdots,\beta_m,\beta_{m+1},\cdots,\beta_{m+k}\}$ of the root system $\Delta$ of $\mathfrak{g}(A)$ is called *fundamental* if it satisfies the following:

(1) $\bar{\Pi}=\{\beta_r\}_{r=1}^{m+k}$ is a linearly independent subset of $\mathfrak{h}^*$;

(2) $\beta_s-\beta_t\notin\Delta\cup\{0\}$ $(1\le s\ne t\le m+k)$;

(3) $\beta_i$ is a real root $(1\le i\le m)$ and $\beta_j$ is a positive imaginary root $(m+1\le j\le m+k)$.

Now, let $(\cdot|\cdot)$ be a fixed standard invariant form on $\mathfrak{g}(A)$ such that $(\alpha_i|\alpha_j)\in\boldsymbol{Z}$ $(1\le i,j\le n)$, where $\{\alpha_i\}_{i=1}^n\subset\Delta$ is the set of all simple roots of $\mathfrak{g}(A)$ (cf. [3, Chap. 2]). For each imaginary root $\beta_j$ $(m+1\le j\le m+k)$, we define $\beta_j^\vee:=\nu^{-1}(\beta_j)\in\mathfrak{h}$, where $\nu:\mathfrak{h}\to\mathfrak{h}^*$ is a linear isomorphism determined by $\langle\nu(h),h'\rangle=(h|h')$ $(h,h'\in\mathfrak{h})$. For real root $\beta_i$ $(1\le i\le m)$, $\beta_i^\vee\in\mathfrak{h}$ has been defined as a dual real root of $\beta_i$, and we know $\beta_i^\vee=2/(\beta_i|\beta_i)\cdot\nu^{-1}(\beta_i)$ (cf. [3, Chap. 5]).

**Proposition 1.1.** *Let $\bar{\Pi}=\{\beta_r\}_{r=1}^{m+k}$ be a fundamental subset of $\Delta$, and put $\bar{A}:=(\bar{a}_{ij})_{i,j=1}^{m+k}$, where $\bar{a}_{ij}=\langle\beta_j,\beta_i^\vee\rangle$. Then, $\bar{A}$ is a symmetrizable GGCM (= generalized GCM). Moreover, $\bar{a}_{ii}=2$ if and only if $\beta_i$ is a real root $(1\le i\le m+k)$.*

Here, $\bar{A}$ is a GGCM means that $\bar{A}$ satisfies the following:

(C1) either $\bar{a}_{ii}=2$ or $\bar{a}_{ii}$ is a non-positive integer;

(C2) $\bar{a}_{ij}$ is a non-positive integer if $i\ne j$;