2. *On the Eigenfunctions for the Sturm-Liouville Equation: Viewed as Functions of the S−L Operator*

By Takao NAMBU

Department of Mathematics, Faculty of Engineering, Kumamoto University

(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1990)

§1. **Introduction.** One of typical elliptic boundary value problems is the following Sturm-Liouville equation (or $S-L$ equation) on a finite interval $a \leq x \leq b$;

$$(1)\qquad \begin{aligned} \mathcal{L}u &= -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+q(x)u=f, \qquad a<x<b, \\ \tau u &= p\frac{du}{d\nu}+\sigma u=g, \qquad x=a,\, b, \end{aligned}$$

where the function p is positive on $a \leq x \leq b$, and $d/d\nu$ indicates the outword normal differentiation at the endpoints $x=a, b$. As is well known, the study of the equation has a long history, and very many useful results, such as the asymptotic behavior of the eigenvalues and eigenfunctions for the operators $\mathcal{L}$ and τ, are known [1, 3, 5]. While promoting a study of some control theoretic problem of one-dimensional parabolic equations, the author posed a question; are the eigenfunctions for $\mathcal{L}$ and τ continuously (in an adequate topology) dependent on the coefficients p, q, and σ? Thus, the eigenfunctions are viewed as functions of $\mathcal{L}$ and τ. Continuous dependence of the eigenvalues on p, q, and σ is well known [1] (see Lemma 1.1 below). As far as the author knows, however, no answer to the above question has been obtained. It is the purpose of the paper to derive an affirmative answer to the question. The result is fundamental, and will be useful, for example, for examining stiffness of feedback control schemes of one-dimensional parabolic systems against small changes of the parameters p, q, and σ.

Set $I=(a, b)$. Let us begin with defining an operator L acting in $L^2(I)$ as follows:

$$Lu=\mathcal{L}u, \qquad u \in \mathcal{D}(L)=\{u \in H^2(I)\,;\ \tau u=0 \ \ x=a, b\}.$$

All norms hereafter will be either $L^2(I)$- or $\mathcal{L}(L^2(I))$- norm unless otherwise indicated. The operator L is clearly self adjoint. It is well known that there is a set of eigenpairs $\{\lambda_n, \varphi_n\}$ such that

(i) $\sigma(L)=\{\lambda_1, \lambda_2, \cdots\}$; $-\infty<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots\to\infty$;

(ii) $L\varphi_n=\lambda_n\varphi_n$, $n \geq 1$; and

(iii) the set $\{\varphi_n\,;\ n\geq 1\}$ forms a CONS in $L^2(I)$.

When $\mathcal{L}$ and τ are perturbed, the resultant operators will be written as

$$\tilde{\mathcal{L}}u=-\frac{d}{dx}\left(\tilde{p}(x)\frac{du}{dx}\right)+\tilde{p}(x)u,$$

and