81. An Improvement of Sufficient Conditions for Starlike Functions

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1. **Introduction.** Let A_p denote the class of functions of the form

(1.1) $$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in N = \{1, 2, 3, \cdots\})$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

A function $f(z) \in A_p$ is said to be p-valently starlike in U if it satisfies

(1.2) $$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0 \qquad (z \in U).$$

We denote by $S^*(p)$ the subclass of A_p consisting of all such functions, and by $S^*(1) = S^*$ when $p = 1$.

For $f(z)$ in the class A_1 when $p = 1$, Singh and Singh [4] have proved

Theorem A. *If $f(z) \in A_1$ satisfies*

(1.3) $$\left| \frac{zf''(z)}{f'(z)} \right| < \frac{3}{2} \qquad (z \in U),$$

then $f(z) \in S^*$.

Also, Miller and Mocanu [2] have showed

Theorem B. *If $f(z) \in A_1$ satisfies*

(1.4) $$\left| \frac{f''(z)}{f'(z)} \right| < 2 \qquad (z \in U),$$

then $f(z) \in S^*$.

In the present paper, we derive an improvement of the above theorems as the special cases of our main result.

2. **Main theorem.** In order to show our main result, we need the following lemma due to Jack [1] (also, by Miller and Mocanu [3]).

Lemma. *Let $w(z)$ be regular in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value in the circle $|z| = r$ at a point z_0, then we can write*

$$z_0 w'(z_0) = k w(z_0),$$

where k is a real number and $k \geq 1$.

Applying the above lemma, we prove

Theorem. *Let $q(z) = p + q_1 z + q_2 z^2 + \cdots \quad (p \in N)$ be analytic in U. If $q(z)$ satisfies*

(2.1) $$\left| q(z) + \frac{zq'(z)}{q(z)} - p \right| < \frac{\sqrt{2}}{8}(5p + 4\sqrt{p} + 4) \qquad (z \in U),$$