10. A Remark on Exponentially Bounded C-semigroups

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1. Introduction. Let X be a Banach space with norm $\|\cdot\|$. We denote by $B(X)$ the set of all bounded linear operators from X into itself.

Let C be an injective operator in $B(X)$. A family $\{S(t); t\geqq 0\}$ in $B(X)$ is called an *exponentially bounded C-semigroup* (hereafter abbreviated to *C-semigroup*) *on* X, if

(1.1) $S(s+t)C=S(s)S(t)$ for s, $t\geqq 0$ and $S(0)=C$,

(1.2) $S(\cdot)$: $[0, \infty)\to X$ is continuous for $x\in X$,

(1.3) there are $M\geqq 0$ and $a\geqq 0$ such that $\|S(t)\|\leqq Me^{at}$ for $t\geqq 0$.

The *generator* A of a C-semigroup $\{S(t); t\geqq 0\}$ on X is defined by

$$(1.4)\qquad \begin{cases} D(A)=\{x\in X;\ \lim_{t\to 0+}(S(t)x-Cx)/t\in R(C)\} \\ Ax=C^{-1}\lim_{t\to 0+}(S(t)x-Cx)/t \quad \text{for } x\in D(A), \end{cases}$$

where $R(C)$ denotes the range of C. It is known ([6, Proposition 1.1]) that

(1.5) A is a closed linear operator in X and $A=C^{-1}AC$.

The purpose of this note is to prove

Theorem 1. *The following statements are equivalent.*

(I) *A is the generator of a C-semigroup on X.*

(II) (a_1) *A is a closed linear operator in X satisfying $C^{-1}AC=A$.*

(a_2) *There exists a Banach space Σ with norm $N(\cdot)$ such that $R(C)\subset\Sigma\subset X$, $\|x\|\leqq M_1N(x)$ for $x\in\Sigma$, $N(x)\leqq M_2\|C^{-1}x\|$ for $x\in R(C)$ and the part of A in Σ is the generator of a semigroup of class (C_0) on Σ, where M_i, $i=1, 2$, are nonnegative constants.*

Corollary 2. *Let A be a closed linear operator in X, $c\in\rho(A)$ (the resolvent set of A) and let $n\geqq 0$ be an integer. Then the following statements are equivalent.*

(I′) *A is the generator of an n-times integrated semigroup on X.*

(II′) *A is the generator of a C-semigroup on X with $C=R(c; A)^n$, where $R(c; A)=(c-A)^{-1}$.*

(III′) *There exists a Banach space Σ with norm $N(\cdot)$ such that $D(A^n)\subset\Sigma\subset X$, $\|x\|\leqq M_1N(x)$ for $x\in\Sigma$, $N(x)\leqq M_2\Sigma_{k=0}^{n}\|A^kx\|$ for $x\in D(A^n)$ and the part of A in Σ is the generator of a semigroup of class (C_0) on Σ, where M_i, $i=1, 2$, are nonnegative constants.*

This corollary improves upon [4, Corollary 5.3].

2. Proofs. Let $\{S(t); t\geqq 0\}$ be a C-semigroup on X satisfying (1.3) and let $b>a$. We define a linear subset Σ of X and a norm $N(\cdot)$ on Σ by

(2.1) $\Sigma=\{x\in X; C^{-1}S(t)x$ is continuous in $t\geqq 0$ and $\lim_{t\to\infty}e^{-bt}\|C^{-1}S(t)x\|=0\}$,

(2.2) $N(x)=\sup_{t\geqq 0}e^{-bt}\|C^{-1}S(t)x\|$ for $x\in\Sigma$,