8. A Counterexample in the Theory of Prehomogeneous Vector Spaces

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1. Let G be a linear algebraic group defined over the complex number field C, (G, ρ, V) a prehomogeneous vector space and Ω the open dense G-orbit in V. (See below for the definitions.) If G is reductive and (G, ρ, V) is regular, then the open subvariety Ω of V is an affine variety [2]. Here the regularity condition is known to be essential, but it seems that the reductivity of G was expected not to be essential. The purpose of this note is to give a counterexample to this expectation.

2. **Prehomogeneous vector spaces.** Let G be as above, $V=C^k$, and $\rho: G\to GL(V)$ a rational representation of G. Such a triplet (G, ρ, V) is called a *prehomogeneous vector space* if V has an open dense G-orbit. (Here and below, we exclusively consider the Zariski topology.) Such an orbit is unique and we shall denote it by Ω. A prehomogeneous vector space (G, ρ, V) is called *regular* if there exists a polynomial function $f(x)=f(x_1, \cdots, x_k)$ on V which satisfies the following two conditions:

(R1) There exists a rational character ϕ of G such that $f(\rho(g)v)=\phi(g)f(v)$ for any $g\in G$ and $v\in V$.

(R2) $$\det\left(\frac{\partial^2 \log f}{\partial x_i \partial x_j}\right)_{1\le i,j\le k}\neq 0 \qquad \text{on } \Omega.$$

3. **Tits system.** Let $G=GL_n(C)$, B be the Borel subgroup of G consisting of upper triangular matrices, T the maximal torus of B consisting of diagonal matrices, $N=N_G(T)$ the normalizer of T in G and $W=N/T$ the Weyl group. Let $\mathfrak{S}_n$ be the symmetric group acting on $\{1, 2, \cdots, n\}$ and $\dot{W}$ the group of permutation matrices in $GL_n(C)$. Then we have natural isomorphisms $\mathfrak{S}_n\simeq \dot{W}\simeq W$, by which we shall identify these three groups. Let S be the set of transpositions $\{(1,2), (2,3), \cdots, (n-1, n)\}$ and

$$w_0=\begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1\end{pmatrix}.$$

Then (G, B, N, S) is a Tits system [1; chapter 4, section 2]. For a subset X of S, let W_X be the subgroup of W generated by X and $G_X=BW_XB$. Every element $w\in W$ can be expressed as $w=s_1s_2\cdots s_a$ $(s_i\in S)$. Define the length $l(w)$ of w to be the minimum of the length a of such an expression. It is known that $l(w)=\dim BwB-\dim B$. If $x, y\in W$ can be expressed as

$$x=s_1s_2\cdots s_a \quad (s_i\in S,\ a=l(x)),$$

and

$$y=s_{i_1}s_{i_2}\cdots s_{i_b} \quad (1\le i_1<i_2<\cdots i_b\le a),$$