

62. The Topological Invariant of Three-manifolds Based on the U(1) Gauge Theory

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In this paper, we shall construct a topological invariants of three-manifolds using a series of representations of the Siegel modular group. We denote by $Sp(2g, \mathbb{Z})$ the Siegel modular group. By a series of projective unitary representations $\rho = \{\rho_g\}_{g \in \mathbb{N}}$ of $Sp(\mathbb{Z}) = \{Sp(2g, \mathbb{Z})\}_{g \in \mathbb{N}}$, we shall mean that $\rho_g: Sp(2g, \mathbb{Z}) \rightarrow PU(V_g)$ is a projective unitary representation on a hermitian vector space V_g for each $g \in \mathbb{N}$.

We shall consider the following two conditions on ρ .

Condition 1. *There exists a hermitian vector space isomorphism $\varphi_g: V_g \otimes V_1 \rightarrow V_{g+1}$ for each $g \in \mathbb{N}$, satisfying*

$\bar{\varphi}_g(\rho_g(X), \rho_g(Y)) = \rho_{g+1}(\iota_g(X, Y))$ for $X \in Sp(2g, \mathbb{Z})$ and $Y \in Sp(2, \mathbb{Z})$, where $\bar{\varphi}_g: PU(V_g) \times PU(V_1) \rightarrow PU(V_{g+1})$ is the homomorphism induced by φ_g and $\iota_g: Sp(2g, \mathbb{Z}) \times Sp(2, \mathbb{Z}) \rightarrow Sp(2g+2, \mathbb{Z})$ is the natural inclusion map.

Condition 2. *There exists an unit vector $v_0^{(1)} \in V_1$, and if we define $v_0^{(g)} \in V_g$ inductively by $v_0^{(g)} = \varphi_{g-1}(v_0^{(g-1)} \otimes v_0^{(1)})$, then $v_0^{(g)}$ is a simultaneous eigenvector of $\rho_g(\mathfrak{p}_g) (\subset PU(V_g))$, where $\mathfrak{p}_g = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}) \mid C=0 \right\}$.*

If we are given a series of projective unitary representations $\rho = \{\rho_g\}_{g \in \mathbb{Z}}$ of $Sp(\mathbb{Z})$ satisfying above Conditions 1 and 2, we can construct a $C/U(1)$ valued topological invariant $W(M)$ for a closed oriented three manifold M in the following manner.

We can represent M by a Heegaard splitting $M = H_g \cup_h (-H_g)$, where H_g is the handle body of genus g , and $h: \sum_g (= \partial H_g) \rightarrow \sum_g$ is an orientation preserving homeomorphism. We fix a basis $\{\lambda_i\}_{i=1}^{2g}$ of $H^1(\sum_g, \mathbb{Z})$ satisfying

$$(1) \quad \begin{cases} \omega(\lambda_i, \lambda_j) = \omega(\lambda_{g+i}, \lambda_{g+j}) = 0 \\ \omega(\lambda_i, \lambda_{g+j}) = \delta_{i,j}, \end{cases} \quad i, j = 1, 2, \dots, g,$$

where ω is the intersection form on \sum_g , and

$$(2) \quad \text{Im } i^* = \bigoplus_{i=1}^g \mathbb{Z} \lambda_i,$$

where $i^*: H^1(\sum_g, \mathbb{Z}) \rightarrow H^1(\sum_g, \mathbb{Z})$ is the homomorphism induced by the inclusion map $i: \sum_g \rightarrow H_g$.

Then we regard $\hat{h} = (h^{-1})^*: H^1(\sum_g, \mathbb{Z}) \rightarrow H^1(\sum_g, \mathbb{Z})$ as an element of $Sp(2g, \mathbb{Z})$ with respect to the basis $\{\lambda_i\}_{i=1}^{2g}$.

Now we define

$$W(M) = c_0^{1-g} \langle \rho_g(\hat{h})v_0^{(g)}, v_0^{(g)} \rangle \quad (\in C/U(1)),$$

and

$$c_0 = \langle \rho_1(J_2)v_0^{(1)}, v_0^{(1)} \rangle \quad (\in C/U(1)),$$