7. *Invariants and Hodge Cycles.* III

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The present work is a continuation of [1] and [2]. Consider a GTAS (Group Theoretic Abelian Scheme) $\pi: A\to V=\Gamma\backslash\mathfrak{D}$. The space

$$H^{(p,p)}(A_\lambda)\cap H^r(A_\lambda, \boldsymbol{Q}), \qquad (r=2p)$$

of Hodge cycles in a generic fibre A_λ is controlled by an invariant theory of G, [1], [2], where G is the $\boldsymbol{Q}$-semisimple algebraic group attached to V. In fact, when A is rigid, the space $H^{(p,p)}(A_\lambda)\cap H^r(A_\lambda,\boldsymbol{Q})$ coincides with the space $H^r(A,\boldsymbol{Q})^G=\Lambda^r(F)^G$ of G-invariant elements in $H^r(A_\lambda,\boldsymbol{Q})$. Here $F=H^1(A_\lambda,\boldsymbol{Q})$. However, this invariant theory is quite different from the classical invariant theory. First, it deals with the exterior product $\Lambda^r(F)$ of the basic representation space F rather than the symmetric product $S^r(F)$ that appears in classical theory. Second, the basic representation (ρ, F) is a very special kind of representation called "rigid polymer type in a chemistry $(\mathfrak{G}, S, S_0)$" that is related to a combinatorics of a finite group $\mathfrak{G}$, [1], [2]. As a result, our invariant theory becomes quite different from the classical one and even the determination of $\dim\Lambda^r(F)^G$, the dimension of the space of invariants, becomes difficult in general, [1], [2]. However, the asymptotic behavior of $\dim\Lambda^r(\mu F)^G$ (as $\mu\to\infty$) can be studied. This will be the goal of the present work. Before going further, we would like to thank Dr. David Weeks, Mrs. Oscar Goldman and Professor Shokichi Iyanaga, M.J.A., for their encouragements.

Let a rigid GTAS $\pi: A\to V$ be of quaternion type so that it corresponds to a polyhedron (polymer) P with a generic fibre A_λ. Then the fibre product (with μ factors) $\pi^{(\mu)}: A\times_V\cdots\times_V A\to V$ corresponds to the polyhedron μP and has a generic fibre $\mu A_\lambda=A_\lambda+\cdots+A_\lambda$ (μ terms). If the basic representation space of the GTAS A is $F=H^1(A_\lambda,\boldsymbol{Q})$, then the representation space of $A\times_V\cdots\times_V A$ is $\mu F=H^1(\mu A_\lambda,\boldsymbol{Q})=F\oplus\cdots\oplus F$. Let $F=X_1+\cdots+X_k$ be the decomposition of F into irreducible pieces. In general, $X_i=X_j$ is permitted for $i\neq j$. However, in the present work, we will be concerned with the case where $X_i\neq X_j$ holds for $i\neq j$. We note that the more general case of $F=\mu_0F_0$ with $\mu_0>1$ can be subsumed under the substitution of μ by $\mu\mu_0$ in our calculation. We recall that $[\Lambda^r(\mu X_1+\cdots+\mu X_k)]=[\Lambda^r(\mu F)]=_{df}\dim\Lambda^r(\mu F)^G\in\boldsymbol{Z}$. This function is only additive on the representation ring of G. We have $\Lambda^r(\mu X_1+\cdots+\mu X_k)\cong\bigoplus_{\sum a_{i,j}=r}\bigotimes_{i,j}\Lambda^{a_{i,j}}(X_i)$, where $1\leq i\leq k$ and $1\leq j\leq\mu$. In the preceding direct sum, those summands with $a_{i,j}\leq 1$ for all i, j are said to be of the *first kind* while those with at least one $a_{i,j}>1$ are said to be of the *second kind*. Thus,

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