58. A Weak Convergence Theorem in Sobolev Spaces with Application to Filippov's Evolution Equations

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1. Introduction. Let $\mathfrak{H}$ be a real separable Hilbert space. A correspondence (=multi-valued mapping) $\Gamma: [0, T]\times\mathfrak{H} \twoheadrightarrow \mathfrak{H}$ is assumed to be given. A double arrow $\twoheadrightarrow$ is used in order to indicate the domain and the range of a correspondence. The compact interval $[0, T]$ is endowed with the usual Lebesgue measure dt. The target of this paper is to establish a sufficient condition which assures the existence of solutions of a multi-valued differential equation of the form:

$$(*) \qquad \dot{x}(t) \in \Gamma(t, x(t)), \qquad x(0)=a,$$

where a is a fixed vector in $\mathfrak{H}$.

In Maruyama [8], I have already presented a solution of this problem in the special case of $\mathfrak{H}=\boldsymbol{R}^l$ by making use of the convenient properties of the weak convergence in the Sobolev space $\mathfrak{W}^{1,2}([0, T], \boldsymbol{R}^l)$ consisting of functions of $[0, T]$ into $\boldsymbol{R}^l$; i.e. if a sequence $\{x_n\}$ in $\mathfrak{W}^{1,2}([0, T], \boldsymbol{R}^l)$ weakly converges to some $x^* \in \mathfrak{W}^{1,2}([0, T], \boldsymbol{R}^l)$, then

$$x_n \to x^* \quad \text{strongly in } \mathfrak{L}^1([0, T], \boldsymbol{R}^l), \quad \text{and}$$
$$\dot{x}_n \to \dot{x}^* \quad \text{weakly in } \mathfrak{L}^2([0, T], \boldsymbol{R}^l).$$

However it is well-known that this property does not hold in the space $\mathfrak{W}^{1,2}([0, T], \mathfrak{H})$ consisting of functions of $[0, T]$ into $\mathfrak{H}$ if $\dim \mathfrak{H}=+\infty$. (Cf. Cecconi [5] pp. 28–29.) We shall first provide a new tool to overcome this difficulty in section 2, and then proceed to the existence theorem for the differential equation $(*)$ in section 3.

2. A convergence theorem in $\mathfrak{W}^{1,p}([0, T], \mathfrak{H})$. We denote by $\mathfrak{H}_s$ (resp. $\mathfrak{H}_w$) the Hilbert space $\mathfrak{H}$ endowed with the strong (resp. weak) topology.

Theorem 1. *Let $\mathfrak{H}$ be a real separable Hilbert space and consider a sequence $\{x_n\}$ in the Sobolev space $\mathfrak{W}^{1,p}([0, T], \mathfrak{H})$ $(p\geqq 1)$. Assume that*

(i) *the set $\{x_n(t)\}_{n=1}^{\infty}$ is bounded (and hence relatively compact) in $\mathfrak{H}_w$ for each $t \in [0, T]$, and*

(ii) *there exists some function $\psi \in \mathfrak{L}^p([0, T], (0, +\infty))$ such that*

$$\|\dot{x}_n(t)\| \leqq \psi(t) \quad a.e.$$

Then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ and some $x^* \in \mathfrak{W}^{1,p}([0, T], \mathfrak{H})$ such that

(a) *$z_n \to x^*$ uniformly in $\mathfrak{H}_w$ on $[0, T]$, and*

(b) *$\dot{z}_n \to \dot{x}^*$ weakly in $\mathfrak{L}^p([0, T], \mathfrak{H})$.*

Proof. (a) To start with, we shall show the equicontinuity of $\{x_n\}$. Since ψ is integrable, there exists some $\delta > 0$ for each $\varepsilon > 0$ such that