57. Convergence Theorems for the Pseudo-Conformally Invariant Nonlinear Schrödinger Equation

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§1. **Introduction.** L^α denotes the space of α-summable function on $\boldsymbol{R}^N$ with the norm $\|\cdot\|_\alpha$. H^s represents the standard Sobolev space of order s on $\boldsymbol{R}^N$. We will use the abbreviation $\|\cdot\|=\|\cdot\|_2$. We put $i=\sqrt{-1}$, $\sigma=2+4/N$, $\partial_t=\partial/\partial t$, $\partial_j=\partial/\partial x_j$ $(j=1,\cdots,N)$, $\nabla=(\partial_1,\cdots,\partial_N)$ and $\Delta=\nabla\cdot\nabla$ (Laplace operator on $\boldsymbol{R}^N$). $(I;L^\alpha)$ denotes the space of continuous functions from a interval $I\subset\boldsymbol{R}$ to L^α with the norm $|\cdot|_{\alpha,\infty,I}=\sup_{t\in I}\|\cdot(t)\|_\alpha$. If $I=\boldsymbol{R}$, we will use $|\cdot|_{\alpha,\infty,\boldsymbol{R}}=|\cdot|_{\alpha,\infty}$. μ denotes the Lebesgue measure on $\boldsymbol{R}^N$. For brevity we write $[f>\eta]=\{x\in\boldsymbol{R}^N;\ f(x)>\eta\}$.

This paper is concerned with the following Cauchy problem for the nonlinear Schrödinger equation:

$$C(p)\qquad \begin{aligned}&2i\partial_t u+\Delta u+|u|^{p-1}u=0,\quad (t,x)\in\boldsymbol{R}\times\boldsymbol{R}^N,\\ &u(0,x)=u_0(x)\quad x\in\boldsymbol{R}^N,\end{aligned}$$

where $1<p<2^*-1$ ($2^*=2N/(N-2)$ if $N\geqq 3$, arbitrary number larger than 2 if $N=1$ and 2). It is well known that for any $u_0\in H^1$, there exist an open interval I in $\boldsymbol{R}$ containing the origin and a unique solution $u_p(t,x)$ of $C(p)$ in $C(I;H^1)$ which satisfies two conservation laws;

$$(1)\qquad \|u_p(t)\|=\|u_0\|,$$

$$(2)\qquad E_{p+1}(u_p)=\|\nabla u_p\|^2-\frac{2}{p+1}\|u_p\|_{p+1}^{p+1}=E_{p+1}(u_0).$$

If $1<p<1+4/N$, u_p exists globally in time, i.e., $I=\boldsymbol{R}$ by (2) and the Gagliardo-Nirenberg inequarity. That is, there is a positive constant $C(p,E_p)$ such that

$$(3)\qquad |\nabla u_p|_{2,\infty},\qquad |u_p|_{p+1,\infty}<C(p,E_p).$$

If $p\geqq 1+4/N$, however, there exist singular solar solutions exploding their L^2 norms of the gradient in finite time (blow-up): Each singular solution $u(t)$ shows that

$$(4)\qquad \lim_{t\to T}\|\nabla u(t)\|=\infty\qquad \text{for some } T\in\boldsymbol{R}.$$

So it can occur that

$$(5)\qquad \limsup_{p\uparrow 1+4/N}|\nabla u_p|_{2,\infty}=\limsup_{p\uparrow 1+4/N}|u_p|_{\sigma,\infty}=\infty.$$

Thus our purpose is to obtain more precise analysis of the behavior of (u_p) as $p\uparrow 1+4/N$ in $C(\boldsymbol{R};L^\sigma)$ (or $C(\boldsymbol{R};H^1)$). We will consider the rescaling function:

$$(6)\qquad u_{p,\lambda}(t,x)=\lambda_p^{N/2}u_p(\lambda_p^2 t,\lambda_p x),$$

where

$$(7)\qquad \lambda_p=1/|u_p|_{\sigma,\infty}^{\sigma/2}\quad (\to 0 \text{ as } p\uparrow 1+4/N).$$