29. An Additive Theory of the Zeros of the Riemann Zeta Function

By Akio FUJII
Department of Mathematics, Rikkyo University

(Communicated by Shokichi IYANAGA, M. J. A., May 14, 1990)

The purpose of the present article is to present an additive theory of the zeros of the Riemann zeta function $\zeta(s)$. The details with some more general results will appear elsewhere.

We recall first the well-known Riemann-von Mangoldt formula for the number $N(T)$ of the zeros of $\zeta(s)$ in $0<\mathrm{Re}\, s<1$, $0<\mathrm{Im}\, s\le T$ (cf. p. 179 and p. 256 of Titchmarsh [8]).

$$\text{(A)}: \qquad N(T)=\frac{1}{2\pi}T\log T-\frac{1+\log 2\pi}{2\pi}T+\frac{7}{8}+O\Big(\frac{1}{T}\Big)+S(T),$$

where $T>T_0$ and $S(T)=(1/\pi)\arg\zeta((1/2)+iT)=O(\log T)$.

Under the Riemann Hypothesis (R.H.), it is well-known that $S(T)=O(\log T/\log\log T)$.

We recall second Landau's theorem on an arithmetic connection of the zeros with a prime number (cf. Landau [7]).

$$\text{(B)}: \qquad \sum_{0<\gamma\le T} x^{\rho}=-\frac{T}{2\pi}\Lambda(x)+O(\log T)$$

for any $x>1$, where $\rho=\beta+i\gamma$ denotes a zero of $\zeta(s)$ and $\Lambda(x)=\log p$, if $x=p^k$, with a prime number p and a positive integer k, and $=0$ otherwise.

Under R.H., this can be improved as follows (cf. Fujii [2] and [6]).

(B′) (Under R.H.): For any $x>1$ and $T>T_0$,

$$\sum_{0<\gamma\le T} x^{(1/2)+i\gamma}=-\frac{T}{2\pi}\Lambda(x)+\frac{x^{(1/2)+iT}\log(T/2\pi)}{2\pi i\log x}+O\Big(\frac{\log T}{\log\log T}\Big).$$

We recall next the following result on an arithmetic connection of the zeros with a rational number (cf. Fujii [1], [2], [3] and [4]). We put $e(x)=e^{2\pi i x}$.

(C) (Under R.H.): Let K be an integer ≥ 1. Then we have

$$\lim_{T\to\infty}\frac{1}{(T/2\pi)^{(1/2)(1+(1/K))}}\sum_{0<\gamma\le T} e\Big(\frac{\gamma}{2\pi K}\log\frac{\gamma}{2\pi e\alpha K}\Big)$$
$$=\begin{cases}-e^{\pi i/4}C\Big(\dfrac{a}{q},K\Big) & \text{if } \alpha=\dfrac{a}{q} \text{ with integers } a \text{ and } q\ge 1,\ (a,q)=1\\ 0 & \text{if } \alpha \text{ is irrational } (>0),\end{cases}$$

where we put

$$C\Big(\frac{a}{q},K\Big)=2\cdot K^{(1/2)(1-(1/K))}\overline{S\Big(\frac{a}{q},K\Big)}(K+1)^{-1}\varphi(q)^{-1}\Big(\frac{a}{q}\Big)^{-1/(2K)}$$

and