17. A Note on Capitulation Problem for Number Fields

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(Communicated by Shokichi IYANAGA, M. J. A., Feb. 13, 1989)

Let F be a finite extension of a finite algebraic number field k and let C_k and C_F denote the ideal class groups of k and F respectively. A subgroup A of C_k is said to *capitulates* in F if $A \to 1$ under the natural homomorphism $C_k \to C_F$. The principal ideal theorem of class field theory states that C_k always capitulates in Hilbert's class field K over k. However, as shown in Heider-Schmithals [1], for some k, C_k capitulates already in a proper subfield M of K: $k \subseteq M \subseteq K$, $M \neq K$. In the present note, we shall give further simple examples of such number fields k for which the capitulation of C_k occurs in a proper subfield M of Hilbert's class field K over k*).

1. Let L be a finite abelian (or nilpotent) extension over k. For each prime number p, let L_p denote the maximal p-extension over k contained in L, and let $C_{k,p}$ be the p-class group of k, i.e., the Sylow p-subgroup of C_k. It is then easy to see that C_k capitulates in L if and only if $C_{k,p}$ capitulates in L_p for every prime number p. Applying this for Hilbert's class field K over k, we see that a number field M such as stated in the introduction exists if and only if there is a prime number p such that $C_{k,p}$ capitulates in a proper subfield F of Hilbert's p-class field K_p over k: $k \subseteq F \subseteq K_p$, $F \neq K_p$. In what follows, we shall find k such that the 2-class group $C_{k,2}$ capitulates in a proper subfield of Hilbert's 2-class field K_2 over k.

2. Let p, p_1, p_2 be three distinct prime numbers such that

i) $p \equiv p_1 \equiv p_2 \equiv 1 \bmod 4$, $(p/p_1) = (p/p_2) = -1$, the brackets being Legendre's symbol, and that

ii) the norm of the fundamental unit of the real quadratic field $k' = \boldsymbol{Q}(\sqrt{p_1 p_2})$ is 1.

Let

$$k = \boldsymbol{Q}(\sqrt{p p_1 p_2}).$$

By Iyanaga [3], p. 12, we know for the real quadratic field k that

iii) the 2-class group $C_{k,2}$ is an abelian group of type $(2, 2)$ and that

iv) the norm of the fundamental unit of k is -1.

Since $[K_2 : k] = |C_{k,2}| = 4$ for Hilbert's 2-class field K_2 over k, we see immediately that

$$K_2 = \boldsymbol{Q}(\sqrt{p}, \sqrt{p_1}, \sqrt{p_2}).$$

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*) The author was informed by Prof. S. Iyanaga, that he had been reminded of the problem of finding such number fields k by Dr. Li Delang at Sichuan University, China. For various aspects of capitulation problem in general, see Miyake [4].