15. *On a Theorem of Landau*

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§ 1. **Introduction.** Let $\rho=\beta+i\gamma$ run over the non-trivial zeros of the Riemann zeta function $\zeta(s)$. Landau [7] has shown that for fixed $x>1$,

$$\sum_{0<\gamma\leqq T} x^{\rho}=-\frac{T}{2\pi}\Lambda(x)+O(\log T),$$

where $T>T_0$, $\Lambda(x)=\log p$ if $x=p^k$ with a prime number p and an integer $k\geqq 1$ and $\Lambda(x)=0$ otherwise. Recently, Gonek [5], [6] has clarified the dependence on x in Landau's theorem as follows:

$$\sum_{0<\gamma\leqq T} x^{\rho}=-\frac{T}{2\pi}\Lambda(x)+O(x\log(2x)\cdot\log\log(3x))+O(x\log(2T))$$
$$+O(\log x\cdot \mathrm{Min}\,(T, x/\langle x\rangle))+O(\mathrm{Min}\,(\log T/\log x, T\log T)),$$

where $T, x>1$ and $\langle x\rangle$ is the distance from x to the nearest prime power other than x itself. On the other hand, in Corollary 3 of [1], the author has refined Landau's theorem under the Riemann Hypothesis as follows; for fixed $x>1$,

$$\sum_{0<\gamma\leqq T} x^{(1/2)+i\gamma}=-\frac{T}{2\pi}\Lambda(x)+\frac{x^{(1/2)+iT}\log(T/2\pi)}{2\pi i\cdot\log x}+O\left(\frac{\log T}{\log\log T}\right).$$

The author has also given in Theorem 1′ of [2] a result on the dependence on x which has been suitable for our applications. The purpose of the present article is to refine all of these results under the Riemann Hypothesis, which we shall assume below. We shall prove the following theorem by improving the author's proof in [1].

Theorem. *For $x>1$ and $T>T_0$, we have*

$$\sum_{0<\gamma\leqq T} x^{(1/2)+i\gamma}=-\frac{T}{2\pi}\Lambda(x)+\sqrt{x}\cdot M(x, T)-\frac{x}{2\pi i}F(x, T)$$
$$+O(x\log(2x))+O\left(\log x\,\mathrm{Min}\left(T, \frac{x}{\langle x\rangle}\right)\right)+O\left(x\sqrt{\frac{\log T}{\log\log T}}\right)$$
$$+O\left(x^{(1/2)+(1/\log\log T)}\cdot\log(2x)\cdot\frac{\log T}{\log\log T}\right),$$

where

$$M(x, T)\equiv\frac{1}{2\pi}\int_1^T x^{it}\log\frac{t}{2\pi}dt$$
$$=\begin{cases}\dfrac{x^{iT}\log(T/2\pi)}{2\pi i\log x}+O\left(\dfrac{1}{\log x}+\dfrac{1}{\log^2 x}\right) & \textit{if } \dfrac{1}{\log T}\ll\log x\\[2ex] O\left(\dfrac{\log T}{\log x}\right) & \textit{if } \dfrac{1}{T}\ll\log x\ll\dfrac{1}{\log T}\\[2ex] O(T\log T) & \textit{if } \log x\ll\dfrac{1}{T}\end{cases}$$