13. A Geometric Study on Systems of First Order Differential Equations

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1. Introduction. Let $J^1(\boldsymbol{R}^n, \boldsymbol{R}^m)$ be the jet space of 1-jets $j^1_x(f)$ of local maps f of $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$. Let $\{x_1, \cdots, x_n\}$ (resp. $\{u_1, \cdots, u_m\}$) be the canonical coordinate system on $\boldsymbol{R}^n$ (resp. $\boldsymbol{R}^m$). Then we can introduce the coordinate system $\{x_1, \cdots, x_n, u_1, \cdots, u_m, \cdots, p^i_j, \cdots\}$ on $J^1(\boldsymbol{R}^n, \boldsymbol{R}^m)$ associated with $\{x_1, \cdots x_n, u_1, \cdots, u_m\}$ given by $p^i_j = \partial u_i / \partial x_j$. Let π_1 (resp. π_2) be the usual projection of $J^1(\boldsymbol{R}^n, \boldsymbol{R}^m)$ onto $\boldsymbol{R}^n$ (resp. $\boldsymbol{R}^m$). In the following we assume that $n = m = 2$ and consider a system of differential equations

$$(E) \qquad \begin{cases} F_1(p) \equiv \alpha_1(x)p^1_1 + \beta_1(x)p^2_1 + \alpha_2(x)p^1_2 + \beta_2(x)p^2_2 = 0, \\ F_2(p) \equiv \gamma_1(x)p^1_1 + \delta_1(x)p^2_1 + \gamma_2(x)p^1_2 + \delta_2(x)p^2_2 = 0 \end{cases}$$

on $J^1(\boldsymbol{R}^2, \boldsymbol{R}^2)$. Denote by $\mathcal{S}(E)$ the set of local solutions of E and set $S(E) = \{j^1_x(f)\,;\, f \in \mathcal{S}(E)$ *and* $x \in$ *the domain of* $f\}$ and $I(E) = \{p \in J^1(\boldsymbol{R}^2, \boldsymbol{R}^2)\,;\, F_1(p) = F_2(p) = 0\}$. Then, in general, we have $I(E) \supset S(E)$.

Let us consider the category $\tilde{C}$ of systems of differential equations E which satisfy the following properties around $p_o \in J^1(\boldsymbol{R}^2, \boldsymbol{R}^2)$:

(1) $I(E) = S(E)$,

(2) $\det \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \neq 0 \quad (i = 1, 2)$,

(3) $(\alpha_2\beta_1 - \alpha_1\beta_2)(\gamma_2\delta_1 - \gamma_1\delta_2)(\beta_1\delta_2 - \beta_2\delta_1) \neq 0$.

Denote by $\mathcal{A}(E)$ the pseudogroup of local transformations ϕ on $\boldsymbol{R}^2$ such that, for any $s \in \mathcal{S}(E)$, if $\phi \circ s$ is defined, then $\phi \circ s \in \mathcal{S}(E)$. $\mathcal{A}(E)$ is called the automorphism pseudogroup of E. Then, according to [2], for any element $E \in \tilde{C}$, we have

Proposition 1.1. *The system of defining equations of* $\mathcal{A}(E)$ *around* $x_o = \pi_1(p_o)$ *is given by*

$$\begin{cases} \partial\phi_1/\partial u_1 = a(x)(\partial\phi_1/\partial u_2) + \partial\phi_2/\partial u_2, \\ \partial\phi_2/\partial u_1 = b(x)(\partial\phi_1/\partial u_2) \end{cases}$$

where $\phi = (\phi_1, \phi_2) \in \mathcal{A}(E)$ *and* $a(x) = (\beta_1\delta_2 - \beta_2\delta_1)^{-1}(\beta_1\gamma_2 - \alpha_2\delta_1 + \alpha_1\delta_2 - \beta_2\gamma_1)$, $b(x) = (\beta_1\delta_2 - \beta_2\delta_1)^{-1}(\alpha_2\gamma_1 - \alpha_1\gamma_2)$.

We set $C = \{E \in \tilde{C}\,;\, a(x)$ *and* $b(x)$ *are constant*$\}$. The purpose of this note is to classify systems of differential equations belonging to C from the geometrical viewpoint using the couple of real numbers (a, b) which is called the structure vector of $E \in C$.

2. Preliminary lemma. Let us consider the 4-dimensional Euclidean space $\boldsymbol{R}^4$ with the canonical coordinate system $\{v_1, v_2, v_3, v_4\}$ and a vector field $W = (av_1 + v_2)(\partial/\partial v_1) + bv_1(\partial/\partial v_2) + (av_3 + v_4)(\partial/\partial v_3) + bv_3(\partial/\partial v_4)$ where a and