92. A Property of Certain Analytic Functions Involving Ruscheweyh Derivatives

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1. **Introduction.** Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty} a_k z^k \qquad (p\in\mathcal{N}=\{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z: |z|<1\}$. For functions $f_j(z)$ ($j=1, 2$) defined by

$$f_j(z)=z^p+\sum_{k=p+1}^{\infty} a_{k,j} z^k, \tag{1.2}$$

we define the convolution $f_1*f_2(z)$ of functions $f_1(z)$ and $f_2(z)$ by

$$f_1*f_2(z)=z^p+\sum_{k=p+1}^{\infty} a_{k,1}a_{k,2} z^k. \tag{1.3}$$

With the convolution above, we define

$$D^{n+p-1}f(z)=\left(\frac{z^p}{(1-z)^{n+p}}\right)*f(z) \qquad (f(z)\in\mathcal{A}(p)), \tag{1.4}$$

where n is any integer greater than $-p$. We note that

$$D^{n+p-1}f(z)=\frac{z^p(z^{n-1}f(z))^{(n+p-1)}}{(n+p-1)!}. \tag{1.5}$$

The symbol D^{n+p-1} when $p=1$ was introduced by Ruscheweyh [5], and the symbol D^{n+p-1} was introduced by Goel and Sohi [3]. Therefore, one called the sympol D^{n+p-1} the Ruscheweyh derivative of $(n+p-1)$th order. It follows from (1.5) that

$$z(D^{n+p-1}f(z))'=(n+p)D^{n+p}f(z)-nD^{n+p-1}f(z). \tag{1.6}$$

Recently, Chen and Lan ([1], [2]) have proved some interesting results of certain analytic functions involving Ruscheweyh derivatives.

2. **A property.** In order to derive our main result, we need the following lemma due to Miller and Mocanu [4].

Lemma. *Let $\phi(u, v)$ be a complex valued function,*

$$\phi: \mathcal{D}\longrightarrow C,\ \mathcal{D}\subset C^2 \qquad (C \text{ is the complex plane}),$$

and let $u=u_1+iu_2$, $v=v_1+iv_2$. Suppose that the function $\phi(u, v)$ satisfies

(i) *$\phi(u, v)$ is continuous in $\mathcal{D}$;*

(ii) *$(1, 0)\in\mathcal{D}$ and $\mathrm{Re}\{\phi(1, 0)\}>0$;*

(iii) *for all $(iu_2, v_1)\in\mathcal{D}$ such that $v_1\leq(-1+u_2^2)/2$, $\mathrm{Re}\{\phi(iu_2, v_1)\}\leq 0$.*

Let $q(z)=1+q_1z+q_2z^2+\cdots$ be regular in $\mathcal{U}$ such that $(q(z), zq'(z))\in\mathcal{D}$ for all $z\in\mathcal{U}$. If

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