90. *Differential Inequalities and Carathéodory Functions*

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Let P be the class of functions $p(z)$ which are analytic in the unit disk $E=\{z: |z|<1\}$, with $p(0)=1$ and $\operatorname{Re} p(z)>0$ in E.

If $p(z)\in P$, we say $p(z)$ a Carathéodory function. It is well known that if $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ is analytic in E and $\operatorname{Re} f'(z)>0$ in E, then $f(z)$ is univalent in E [2, 7].

Ozaki [6, Theorem 2] extended the above result to the following:

If $f(z)$ is analytic in a convex domain D and

$$\operatorname{Re}(e^{i\alpha}f^{(p)}(z))>0 \qquad \text{in } D$$

where α is a real constant, then $f(z)$ is at most p-valent in D.

This shows that if $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$ is analytic in E and

$$\operatorname{Re} f^{(p)}(z)>0 \qquad \text{in } E,$$

then $f(z)$ is p-valent in E.

Nunokawa [3] improved the above result to the following:

Let $p\geqq 2$. If $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$ is analytic in E and

$$|\arg f^{(p)}(z)|<\frac{3}{4}\pi \qquad \text{in } E,$$

then $f(z)$ is p-valent in E.

Definition. Let $F(z)$ be analytic and univalent in E and suppose that $F(E)=R$. If $f(z)$ is analytic in E, $f(0)=F(0)$, and $f(E)\subset R$, then we say that $f(z)$ is subordinate to $F(z)$ in E, and we write

$$f(z)\prec F(z).$$

In this paper, we need the following lemmata.

Lemma 1. *If $p(z)$ is analytic in E, with $p(0)=1$ and*

$$\operatorname{Re}(p(z)+zp'(z))>\beta \qquad \text{in } E,$$

where $\beta<1$, then we have

$$\operatorname{Re} p(z)>(1-\beta)\log\frac{4}{e}+\beta \qquad \text{in } E. \tag{1}$$

Proof. Let us put

$$g(z)=\frac{1}{1-\beta}(p(z)+zp'(z)-\beta)$$
$$=\frac{1}{1-\beta}((zp(z))'-\beta).$$

Then we have

$$g(z)\in P.$$

This shows that