82. *Mordell-Weil Lattices and Galois Representation.* III

By Tetsuji SHIODA
Department of Mathematics, Rikkyo University

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1989)

5. Galois representation arising from the Mordell-Weil lattices. To explain the basic idea, let us consider the "elementary" situation. Let E be an elliptic curve defined over $\boldsymbol{Q}(t)$, t being a variable over $\boldsymbol{Q}$, and let $f: S \to \boldsymbol{P}^1$ be its Kodaira-Néron model, which is an elliptic surface defined over $\boldsymbol{Q}$. Assume as before that f is not smooth. Letting $\bar{\boldsymbol{Q}}$ be the algebraic closure of $\boldsymbol{Q}$ and $K=\bar{\boldsymbol{Q}}(t)$, consider the Mordell-Weil group of K-rational points $E(K)$. Obviously the Galois group $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ acts on $E(K)$, and it makes the height pairing stable. Thus we have the Galois representation on the Mordell-Weil lattice $E(K)/(\mathrm{tor})$ or $E(K)^0$: let

$$\rho: \ \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}) \longrightarrow \mathrm{Aut}(E(K)^0). \tag{5.1}$$

There arises a natural question:

Question 5.1. (1) *How big can* $\mathrm{Im}(\rho)$ *be? and:*

(2) *How small can* $\mathrm{Im}(\rho)$ *be?*

The interest of the first question is obvious. The second one is also interesting, because if the image of ρ is trivial, then we have $E(\bar{\boldsymbol{Q}}(t))=E(\boldsymbol{Q}(t))$ so that the rank of E over $\boldsymbol{Q}(t)$ can be relatively big. The intermediate case can be also of some interest (e.g. [5]).

Suppose, for instance, that $S\otimes\bar{\boldsymbol{Q}}$ is a rational elliptic surface without reducible fibres. Then, by Theorem 2.1, the Mordell-Weil lattice is the root lattice E_8, and hence we have

$$\rho: \ \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}) \longrightarrow \mathrm{Aut}(E_8)=W(E_8). \tag{5.2}$$

The Hasse zeta function of the surface S over $\boldsymbol{Q}$ is given by

$$\zeta(S/\boldsymbol{Q}, s)=\zeta(s)\zeta(s-1)^2\zeta(s-2)L(\rho, s-1) \tag{5.3}$$

(up to finitely many Euler factors) where $L(\rho, s)$ is the Artin L-function attached to ρ and $\zeta(s)$ is the Riemann zeta function.

Now the first question asks: is it possible to have $\mathrm{Im}(\rho)=W(E_8)$ for some $E/\boldsymbol{Q}(t)$? We can affirmatively answer this question (Theorem 7.1) and its variant for E_7, E_6, etc. Thus we obtain infinitely many Galois extensions of $\boldsymbol{Q}$ with Galois group $W(E_8)$, having a natural representation ρ on the lattice E_8. Since $W(E_8)$ contains a subgroup H of index 2 such that $H/\{\pm 1\}$ is a simple group ([1, Ch. 6]), $L(\rho, s)$ is essentially of non-abelian type.

Our results also answer the question, first remarked by Weil [9, p. 558] and then studied by Manin [4, Ch. 4] in more detail, concerning the image of the Galois representation arising from the 27 lines on a smooth cubic surface or more generally from the exceptional curves on a Del Pezzo surface (cf. Remark 6.3).