72. Iwasawa's λ-invariants of Certain Real Quadratic Fields

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We studied Greenberg's conjecture (cf. [3]) on real quadratic case in previous papers [1] and [2]. Two natural numbers n_1 and n_2 were defined in [1]. We treated the case $n_1<n_2$ in [1] and the case $n_1=n_2=2$ in [2]. In this paper, we shall make further investigation in the case $n_1=n_2=2$.

Let k be a real quadratic field with class number h, p an odd prime number which splits in $k/\boldsymbol{Q}$ and

$$k=k_0\subset k_1\subset\cdots\subset k_n\subset\cdots\subset k_\infty$$

the cyclotomic $\boldsymbol{Z}_p$-extension with Galois group $G(k_\infty/k)=\overline{\langle\sigma\rangle}$. Let $p=\mathfrak{p}\mathfrak{p}'$ be the prime factorization of p in k and $\mathfrak{p}_n$ (resp. $\mathfrak{p}_n'$) the unique prime ideal of k_n lying above $\mathfrak{p}$ (resp. $\mathfrak{p}'$). Let A_n be the p-primary part of the ideal class group of k_n and put $D_n=\langle\mathrm{cl}(\mathfrak{p}_n)\rangle\cap A_n$, $B_n^{(r)}=\{a\in A_n\,|\,a^{\sigma_r-1}=1\}$ for $0\leqq r\leqq n$ where $\sigma_r=\sigma^{p^r}$. We put $B_n=B_n^{(0)}$. The norm maps $N_{n,m}\colon k_n\to k_m$ $(0\leqq m\leqq n)$ are applied to A_n, the unit group E_n of k_n and etc.

From now on we assume that $n_1=n_2=2$. (See [1] on the definition of n_1 and n_2.) In this case, the following lemma which was proved in [1] and [3] is fundamental.

Lemma 1. *Let k be a real quadratic field and p an odd prime number which splits in $k/\boldsymbol{Q}$. Assume that*

(1) $n_1=n_2=2$, *and*

(2) $A_0=1$.

Then, $|B_n|=p$, $E_0\cap N_{n,0}(k_n^\times)=E_0^{p^{n-1}}$, *and* $(B_n:D_n)=(E_0\cap N_{n,0}(k_n^\times):N_{n,0}(E_n))$ *for all* $n\geqq 1$. *Futhermore,* $\mu_p(k)=\lambda_p(k)=0$ *if and only if* $D_n\neq 1$ *for some* $n\geqq 1$.

Now we assume that $D_r=1$ for some $r\geqq 1$ and choose $\alpha_r\in k_r$ such that $\mathfrak{p}_r'^h=(\alpha_r)$. We define the natural number $n_1^{(r)}$ by

$$\mathfrak{p}^{n_1^{(r)}}\,\|\,(N_{r,0}(\alpha_r)^{p-1}-1).$$

Since $N_{r,0}(E_r)=E_0^{p^r}$ from Lemma 1, $n_1^{(r)}$ is uniquely determined under the condition $r+1\leqq n_1^{(r)}\leqq r+2$. For $k^*=k(e^{2\pi\sqrt{-1}/p})$, we have the following result.

Proposition. *Let k and p be as in Lemma* 1. *In addition to the assumptions* (1) *and* (2) *of Lemma* 1, *we assume that*

(3) $\lambda_p^-(k^*)=1$, *and*

(4) $D_r=1$ *for some* $r\geqq 1$.

Then, $D_{r+1}\neq 1$ *is and only if* $n_1^{(r)}=r+1$. *In particular,* $\mu_p(k)=\lambda_p(k)=0$ *if* $n_1^{(r)}=r+1$.

For the Proof of Proposition, we need some lemmas. Let K_n denote