37. *Properties of Certain Analytic Functions*

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1. Introduction. Let $A(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty} a_{p+k}\, z^{p+k} \quad (p \in N=\{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the unit disk $U=\{z : |z|<1\}$.

Further, we define a function $F_\lambda(z)$ by

$$F_\lambda(z)=(1-\lambda)f(z)+\lambda z f'(z) \tag{1.2}$$

for $\lambda \geqq 0$ and $f(z) \in A(p)$. In the present paper, we derive some properties of functions in the class $A(p)$, and of the function $F_\lambda(z)$ defined by (1.2).

2. Main results. We begin with the statement of the following lemma due to Miller [1].

Lemma. *Let $\phi(u, v)$ be a complex valued function such that*

$$\phi : D \to C, \quad D \subset C \times C \ (C \text{ is the complex plane}),$$

and let $u=u_1+iu_2$, $v=v_1+iv_2$. Suppose that the function $\phi(u, v)$ satisfies

(i) *$\phi(u, v)$ is continuous in D,*

(ii) *$(1, 0) \in D$ and $\mathrm{Re}\{\phi(1, 0)\}>0$,*

(iii) *for all $(iu_2, v_1) \in D$ such that $v_1 \leqq -(1+u_2^2)/2$, $\mathrm{Re}\{\phi(iu_2, v_1)\} \leqq 0$.*

Let $p(z)=1+p_1 z+p_2 z^2+\cdots$ be regular in the unit disk U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$\mathrm{Re}\{\phi(p(z), zp'(z))\}>0 \quad (z \in U),$$

then $\mathrm{Re}\{p(z)\}>0$ $(z \in U)$.

Applying the above lemma, we prove

Theorem 1. *Let a function $f(z)$ defined by* (1.1) *be in the class $A(p)$. If*

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\}>\alpha \quad \left(0 \leqq \alpha<\frac{p\,!}{(p-j)\,!}\,;\ z \in U\right),$$

then we have

$$\mathrm{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}>\frac{1}{(p-j+1)\,!}\,\frac{(p-j+1)\,!\,2\alpha+p\,!}{2(p-j)+3} \quad (z \in U),$$

where $1 \leqq j \leqq p$.

Proof. We define the function $p(z)$ by

$$\frac{(p-j+1)\,!}{p\,!}\,\frac{f^{(j-1)}(z)}{z^{p-j+1}}=\beta+(1-\beta)p(z) \tag{2.1}$$

with $\beta=\dfrac{(p-j+1)\,!\,2\alpha+p\,!}{p\,!\{2(p-j)+3\}}$. Then $p(z)=1+p_1 z+p_2 z^2+\cdots$ is regular in U. Differentiating both sides in (2.1), we obtain