31. Yang-Mills-Higgs Fields and Harmonicity of Limit Maps

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(Communicated by Kunihiko KODAIRA, M. J. A., April 12, 1989)

Consider a connection A and a Higgs field Φ on the trivial $SU(2)$ bundle over $\boldsymbol{R}^3$, the Euclidean 3-space. A configuration (A, Φ) is called a Yang-Mills-Higgs field if it is a critical point of the action integral $\mathcal{Y}(A, \Phi)=\int_{\boldsymbol{R}^3}\{|F_A|^2+|\nabla_A\Phi|^2\}d^3x$ ($F_A=dA+[A, A]$ and $\nabla_A\Phi+d\Phi+[A, \Phi]$ denote the curvature of A and the covariant derivative of Φ, respectively).

Yang-Mills-Higgs field satisfies the Euler-Lagrange equations $d_A{*F}+[\Phi, {*\nabla_A\Phi}]=0$, $d_A({*\nabla_A\Phi})=0$.

The infinity condition on Higgs fields Φ: $|\Phi|(x)\to 1(|x|\to\infty)$ should be posed in order to avoid the trivial case. Then, for each (A, Φ) the degree of the normalized Higgs field at the infinity 2-sphere $\Phi/|\Phi|: S^2_\infty\to S^2\subset \mathfrak{su}(2)$ defines $k\in \boldsymbol{Z}$, called the charge.

A configuration (A, Φ) with finite $\mathcal{Y}(A, \Phi)$ satisfying Bogomolnyi equations, $\nabla_A\Phi=\pm{*F_A}$, yields a Yang-Mills-Higgs field. We call such a Yang-Mills-Higgs field a magnetic monopole.

Yang-Mills-Higgs fields correspond to 4-dimensional Yang-Mills connections and magnetic monopoles to (anti-)instantons.

Like the moduli space of instantons, the moduli space of charge k monopoles is variously considered. It turns out that the moduli space M_k is a complete hyperkähler manifold ([2]). The twistor formalism was applied by Hitchin and monopoles were transferred into holomorphic structures on a certain complex vector bundle over the space $G(\boldsymbol{R}^3)$ of all oriented lines in $\boldsymbol{R}^3$ and it was further shown that monopoles are interpreted as solutions to Nahm's equations ([5], [6]). By using these, Donaldson proved that M_k is in a one-to-one correspondence to a complex manifold $\mathcal{R}_k$ of all holomorphic maps $f: \boldsymbol{CP}^1\to\boldsymbol{CP}^1$, $f(\infty)=0$, of degree k([3]).

This observation is considered as presentation of a correspondence between the two different variational objects: Yang-Mills-Higgs fields and harmonic maps, because every holomorphic map is harmonic. A harmonic map $f: S^2\to X$ is critical for the energy functional $\mathcal{E}(f)=\int_{S^2}|df|^2d\sigma$ ([4]).

In this paper we obtain the following phenomenon which gives a more direct representation of Yang-Mills-Higgs fields into harmonic maps by using the limis of Higgs fields at infinity.

Theorem 1. *Let* $[(A, \Phi)]$ *be a gauge equivalence class of* $SU(2)$ *Yang-Mills-Higgs field of* $\mathcal{Y}(A, \Phi)<\infty$ *with the asymptotical condition*