29. *Hypoellipticity and Existence of Periodic Solutions on T^d*

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1. Introduction. In this paper we shall study the solvability and the regularity of linear and semilinear equations on T^d which are not necessarily elliptic. We know, by examples, that the regularity and the solvability of such operators are often expressed by diophantine conditions, such as a Siegel condition, etc. Hence it is interesting to seek operators for which necessary and sufficient conditions for the solvability or the regularity are equivalent to Siegel-type diophantine conditions.

Roughly speaking, such operators are characterized by (A. 1) which follows (cf. Remark 2. 4). Then we study them in connection with the global hypoellipticity and solvability on T^d, and the existence of periodic solutions for semilinear equations on T^d whose ratio of periods is not necessarily rational. In the former case, these operators clearly reveal the difference of the hypoellipticity and the global hypoellipticity which is still in question (cf. [1], [2]). In the latter case, the general theory does not work because of small denominators (cf. [4]). We can show the existence of solutions in this case.

§2. Notations and results. Let $T^d=R^d/(2\pi)Z^d$ be a d-dimensional torus. We denote the variables in T^d by $x=(x_1, \cdots, x_d)$. For a multi-index $\alpha=(\alpha_1, \cdots, \alpha_d)\in N$, $N=\{0,1,2,\cdots\}$ we set $D^\alpha=(-i\partial/\partial x_1)^{\alpha_1}\cdots(-i\partial/\partial x_d)^{\alpha_d}$. For $\beta=(\beta_1, \cdots, \beta_d)\in Z^d$ and $z=(z_1, \cdots, z_d)\in C^d$ we set $\langle\beta\rangle=1+|\beta|$, $|\beta|=|\beta_1|+\cdots+|\beta_d|$ and $\|z\|=(|z_1|^2+\cdots+|z_d|^2)^{1/2}$. A function $f(x)$ of $x\in R^d$ is identified with a function on the torus T^d if $f(x+2\pi\eta)=f(x)$ for all $x\in R^d$ and $\eta\in Z^d$. We denote the sets of distributions and infinitely differentiable functions on T^d, respectively by $\mathcal{D}'(T^d)$ and $C^\infty(T^d)$.

For $s\geq 1$, we define a Gevrey class $G^s(T^d)$ of order s by

$$(2.1)\qquad G^s(T^d)=\{f=\sum_{\gamma\in Z^d} f_\gamma e^{i\gamma x}\in C^\infty(T^d)\,|\,{}^\exists\varepsilon>0,\ {}^\exists K>0 \text{ such that } |f_\gamma|\leq K\exp(-\varepsilon|\gamma|^{1/s}),\ \forall\gamma\in Z^d\}.$$

We remark that the above definition of a Gevrey class agrees with the usual one.

Let $m\geq 1$ be an integer, and let $p_m(\eta)$ be a polynomial of degree m, $p_m(\eta)=\sum_{|\alpha|=m} a_\alpha\eta^\alpha$ with $a_\alpha\in C$, and let $b(x,D)$ be a classical pseudo-differential operator on T^d of order $m-1$. We denote by $b(x,\eta)\in C^\infty(T^d\times R^d)$ the symbol of $b(x,D)$. We assume that $\langle\eta\rangle^{1-m}b(x,\eta)$ is uniformly in $G^s(T^d)$, namely we can take ε and K in (2.1) independent of $\eta\in Z^d$. We consider the following operator on T^d: