15. On the Erdös-Turán Inequality on Uniform Distribution. II

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This is continued from [1].

2. To prove Theorem 1 we need three lemmas.

Lemma 1. *Let a function f satisfy the right Lipschitz condition on $\boldsymbol{R}$ with constant L, and let Δ be a closed interval. Set $\delta=\|f\|/2L$, where $\|f\|$ denotes the supremum norm of f on Δ. Then there exists a real number a such that either*

(5) $$f(x+a)\geqq L(x+\delta) \qquad \text{for all } x<\delta$$

or

(6) $$f(x+a)\leqq L(x-\delta) \qquad \text{for all } x>-\delta.$$

Proof. By the assumption, it follows that f is a function of bounded variation on every closed interval. Hence, both limit values $f(x+)$ and $f(x-)$ exist for every $x\in\boldsymbol{R}$. Moreover, we have

(7) $$f(x+)\leqq f(x)\leqq f(x-) \qquad \text{for all } x\in\boldsymbol{R}.$$

Indeed, since f satisfies the right Lipschitz condition with constant L, we have

$$f(x+t)-Lt\leqq f(x)\leqq f(x-t)+Lt$$

for all $x\in\boldsymbol{R}$ and $t>0$. Passing to the limit in these inequalities as $t\to 0+$ we obtain (7).

Let us consider f on the closed interval Δ. Then from (7), it follows that there exists a point $b\in\Delta$ such that either $\|f\|=f(b-)$ or $\|f\|=-f(b+)$. Now set

(8) $$a=\begin{cases} b-\delta & \text{if } \|f\|=f(b-),\\ b+\delta & \text{if } \|f\|=-f(b+).\end{cases}$$

We shall prove that the real number a defined by (8) satisfies the requirement of the lemma.

Suppose first that $\|f\|=f(b-)$. Then from the definition of δ, we conclude that $f(b-)=2L\delta$. Now choose two real numbers y and t with $y<t<b$. Since f satisfies the right Lipschitz condition on $\boldsymbol{R}$ with constant L,

$$f(y)\geqq f(t)-L(t-y).$$

Passing to the limit in this inequality as $t\to b-$ we obtain

(9) $$f(y)\geqq f(b-)-L(b-y)=2L\delta-L(b-y).$$

Now let $x<\delta$. Then (8) implies that $x+a<b$. Hence, we can apply (9) with $y=x+a$. Thus, we arrive at

$$f(x+a)\geqq 2L\delta-L(b-a-x)=2L\delta-L(\delta-x)=L(x+\delta),$$