14. *Quadratic Conservatives of Linear Symplectic System*

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1. Introduction. The problem treated in this paper is explained as follows. Let G and g denote the $2N$-dimensional symplectic group $Sp(N, \boldsymbol{R})$ and its Lie algebra, respectively:

$$G=\{A \in M(2N, \boldsymbol{R}) \mid A'JA=J\}, \qquad g=\{X \in M(2N, \boldsymbol{R}) \mid X'J+JX=0\},$$

where $J=\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ and dash denotes matrix transpose. Our problem is to establish an algebraic approach to finding the quadratic form

(1) $$f_S(x)=x'Sx/2, \qquad S'=S,$$

conserved along any solution of the linear recurrence on $\boldsymbol{R}^N$

(2) $$x_{t+1}=Ax_t,$$

where A is an arbitrary element of G. The whole of conservatives given by (1) forms a Lie algebra with respect to the Poisson bracket [1]. This problem aims at finding economic conservation laws [2, 3] of a discrete economic growth model, though it seems trivial at a glance.

2. Linear space Ξ. In this preliminary section, we introduce a linear space Ξ of all matrices commuting with A, and it is proved that one of its subspaces is Lie algebra isomorphic to the whole of quadratic conservatives given by (1).

Now, (1) is conserved along any solution of (2), if and only if A and JS commute. Then, the whole of quadratic conservatives is identified with the following linear space of all coefficient matrices:

$$\Omega=\{S \in M(2N, \boldsymbol{R}) \mid [A, JS]=0,\ S'=S\},$$

where $[A, B]=AB-BA$. Ω forms a Lie algebra with respect to the bracket

(3) $$\langle S, T\rangle=SJT-TJS,$$

which is a representation of the Poisson bracket on Ω. Apart from looking into Ω directly, we introduce a linear space Ξ of all matrices that commute with A;

$$\Xi=\{L \in M(2N, \boldsymbol{R}) \mid [A, L]=0\}.$$

We define two linear mappings $\eta: \Xi \to \Xi$ and $\sigma: \Xi \to \Omega$ by

(4) $$\eta(L)=JL'J, \qquad \sigma(L)=J(L+\eta(L))/2=(JL+(JL)')/2.$$

Lemma 1. $\eta^2=id.$, $\eta(\Xi)=\Xi$.

Proof. Let $L \in \Xi$. Then, it follows from direct calculation that $\eta^2(L)=L$ and $[\eta(A), \eta(L)]=\eta([A, L])=0$. Since A is symplectic, we have $\eta(A)=-A^{-1}$ and accordingly $[A, \eta(L)]=0$, which means $\eta(\Xi)\subset\Xi$. This together with $\eta^2=id.$ leads to $\eta(\Xi)=\Xi$.

The lemma shows that η is an involution map on Ξ. Then, η has two eigenvalues ± 1 and Ξ is a direct sum of the two. That is, $\Xi=\Theta \dot{+} \Phi$, where