108. Two-Phase Stefan Problems for Parabolic-Elliptic Equations

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1. Statement of the problem. Let us consider a two-phase Stefan problem described as follows: Find a function $u=u(t,x)$ on $Q=(0,T)\times(0,1)$, $0<T<\infty$, and a curve $x=l(t)$, $0<l<1$, on $[0,T]$ such that

$$(0.1)\quad \rho(u)_t-a(u_x)_x+h(t,x)=\begin{cases} f_0 & \text{in } Q_l^+,\\ f_1 & \text{in } Q_l^-,\end{cases}$$

$$h(t,x)\in g(u(t,x)) \quad \text{for a.e. } (t,x)\in Q,$$

$$Q_l^+=\{(t,x)\,;\,0<t<T,\ 0<x<l(t)\},\ Q_l^-=\{(t,x)\,;\,0<t<T,\ l(t)<x<1\},$$

$$(0.2)\quad \begin{cases} u(t,l(t))=0 \quad \text{for } 0\leqq t\leqq T,\\ l'(t)=-a(u_x(t,l(t)-))+a(u_x(t,l(t)+)) \quad \text{for a.e. } t\in(0,T),\ l(0)=l_0,\end{cases}$$

$$(0.3)\quad \rho(u(0,x))=v_0(x) \quad \text{for } 0\leqq x\leqq 1,$$

$$(0.4)\quad \begin{cases} a(u_x(t,0+))\in\partial b_0^t(u(t,0)) & \text{for a.e. } t\in(0,T),\\ -a(u_x(t,1-))\in\partial b_1^t(u(t,1)) & \text{for a.e. } t\in(0,T),\end{cases}$$

where $\rho: R\to R$ is a non-decreasing function and $a: R\to R$ is a continuous function; $g(\cdot)$ is a maximal monotone graph in $R\times R$; f_0, f_1 are functions on Q; l_0 is a number with $0<l_0<1$ and v_0 is a function on the interval $(0,1)$; for $i=0,1$, b_i^t is a proper l.s.c. convex function on R and ∂b_i^t is its subdifferential. We note that the expression (0.4) includes various boundary conditions such as Dirichlet, Neumann and Signorini boundary conditions.

In the case when $a(r)=r$ and $g(r)\equiv 0$, Crowley [2] proved the uniqueness of solution to the multi-dimensional problem in a weak formulation and recently Cannon-Yin [1] established an existence result for (0.1)–(0.4) under the additional restriction that ρ is strictly increasing in R.

In this paper, we suppose that ρ is non-decreasing, and we are very interested in the additional heat source term $g(u)$, which causes unusual behavior of the solution $\{u, l\}$. For instance, as is seen from the following example, $\Omega_0(t):=\{x\in[0,1]\,;\,u(t,x)=0\}$ has positive linear measure. This region $\Omega_0(t)$ is called the mushy region and was analized by M. Bertsch, P. de Mottoni and L. A. Peletier [1, 2].

Example. Suppose that $T=3$,

$$\rho(r)=\begin{cases} r-1 & \text{for } r>1,\\ 0 & \text{for } |r|\leqq 1,\\ r+1 & \text{for } r<-1,\end{cases}\qquad a(r)=r,$$

$$g(r)=\mathrm{sign}(r)=\begin{cases} 1 & \text{for } r>0,\\ [-1,1] & \text{for } r=0,\\ -1 & \text{for } r<0,\end{cases}\qquad f_0=f_1=0,$$